DEPTH-INTEGRATED, NON-HYDROSTATIC MODEL WITH GRID NESTING FOR TSUNAMI GENERATION, PROPAGATION, AND RUNUP

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By

Yoshiki Yamazaki

Dissertation Committee:

Kwok Fai Cheung, Chair
Gerard J. Fryer
Geno Pawlak
Ian N. Robertson
John C. Wiltshire
We certify that we have read this dissertation and that, in our opinion, it is satisfactory in scope and quality as a dissertation for the degree of Doctor of Philosophy in Ocean and Resources Engineering.

DISSERTATION COMMITTEE

__________________________________________
Chairperson

__________________________________________

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ABSTRACT

This dissertation describes the formulation, verification, and validation of a dispersive wave model with a shock-capturing scheme, and its implementation for basin-wide evolution and coastal runup of tsunamis using two-way nested computational grids. The depth-integrated formulation builds on the nonlinear shallow-water equations and utilizes a non-hydrostatic pressure term to describe weakly dispersive waves. The semi-implicit, finite difference solution captures flow discontinuities associated with bores or hydraulic jumps through a momentum conservation scheme, which also accounts for energy dissipation in the wave breaking process without the use of an empirical model. An upwind scheme extrapolates the free surface elevation instead of the flow depth to provide the flux in the momentum and continuity equations. This eliminates depth extrapolation errors and greatly improves the model stability, which is essential for computation of energetic breaking waves and runup.

The vertical velocity term associated with non-hydrostatic pressure also describes tsunami generation and transfer of kinetic energy due to dynamic seafloor deformation. A depth-dependent Gaussian function smoothes bathymetric features smaller than the water depth to improve convergence of the implicit, non-hydrostatic solution. A two-way grid-nesting scheme utilizes the Dirichlet condition of the non-hydrostatic pressure and both the velocity and surface elevation at the grid interface to ensure propagation of dispersive waves and discontinuities through computational grids of different resolution. The inter-grid boundary can adapt to topographic features to model wave transformation processes at optimal resolution and computational efficiency.

The computed results show very good agreement with data from previous laboratory experiments for wave propagation, transformation, breaking, and runup over a wide range.
of conditions. The present model is applied to the 2009 Samoa Tsunami for demonstration and validation. These case studies confirm the validity and effectiveness of the present modeling approach for tsunami research and impact assessment. Since the numerical scheme to the momentum and continuity equations remains explicit, the implicit non-hydrostatic solution is directly applicable to existing nonlinear shallow-water models.
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CHAPTER 1
INTRODUCTION

Tsunamis generated by earthquakes are typically modeled as shallow-water waves with a spherical coordinate system for their propagation around the globe. The finite difference method, owing to its simplicity in formulation and ease of implementation, is widely used in the solution schemes of the depth-integrated governing equations. These explicit schemes provide an efficient approach to describe the tsunami evolution process from generation, propagation, to runup through a series of two-way nested grids (e.g., Liu et al., 1995b; Goto et al., 1997; Titov and Synolakis, 1998; and Wei et al., 2008). Consistent with the shallow-water approach, the initial tsunami wave assumes the vertical component of the seafloor deformation due to earthquake rupture (Kajiura, 1970). This static sea surface deformation simplifies the tsunami generation mechanism to provide a practical method for reconstruction of real-field tsunamis. Recent studies, however, suggest wave dispersion and breaking as well as dynamic seafloor deformation could have non-negligible effects on the tsunami evolution and runup processes (Ohmachi et al., 2001; Horrillo et al., 2006; Wei et al., 2006; and Song et al., 2008).

The commonly used finite difference schemes for the nonlinear shallow-water equations are non-conservative leading to volume loss and energy dissipation as the wave steepness increases and the flow approaches discontinuous. This turns out to be a crucial modeling issue when tsunami bores develop nearshore and the results become grid-size dependent. The finite volume method based on the conservative form of the nonlinear shallow-water equations captures flow discontinuities and conserves flow volume through a Riemann solver (e.g., Dodd, 1998; Zhou et al., 2001; Wei et al., 2006; Wu and Cheung, 2008; and George, 2010). This approach approximates breaking waves as bores or hydraulic jumps through conservation of momentum and accounts for energy dissipation across flow
discontinuities without the use of empirical relations. Stelling and Duinmeijer (2003) developed an equivalent shock-capturing scheme for finite difference shallow-water models. This momentum-conserved advection scheme, which is derived from the conservative formulation of the momentum equations, produces comparable results as a Riemann solver.

The nonlinear shallow-water equations are hydrostatic and are unable to describe wave dispersion. Through a classical Boussinesq model, Horrillo et al. (2006) showed wave dispersion in basin-wide tsunami propagation results in a sequence of trailing waves that might have significant effects on coastal runup. Boussinesq-type models, however, involve a complex equation system with high-order dispersive terms that might not be amenable to the grid-nesting schemes commonly used in tsunami modeling. In contrast, Casulli (1995) proposed an alternative approach to model dispersive waves that decomposes the pressure into hydrostatic and non-hydrostatic components. The explicit hydrostatic solution is determined first and then an implicit solution is obtained for wave dispersion associated with the non-hydrostatic pressure. Researchers have implemented this approach in the formulation of full three dimensional, multi-layer three-dimensional, and depth-integrated two-dimensional models (e.g., Mahadevan et al., 1996a; Casulli and Stelling, 1998; Stansby and Zhou, 1998; Stelling and Zijlema, 2003; and Walters, 2005). The dispersive term is described with the first derivative of the non-hydrostatic pressure in both three-dimensional and two-dimensional models. The depth-integrated governing equations are relatively simple and analogous to the nonlinear shallow-water equations with the addition of a vertical momentum equation and non-hydrostatic terms in the horizontal momentum equations. The depth-integrated, non-hydrostatic models of Stelling and Zijlema (2003) and Walters (2005) produce comparable or better results in comparison to the classical Boussinesq equations.
Zijlema and Stelling (2008) presented a multi-layer, non-hydrostatic formulation with a momentum conservation scheme for the advective terms and an upwind approximation in the continuity equation, and derived semi-implicit schemes for both the hydrostatic and non-hydrostatic solutions. Their two-layer model can handle wave breaking without the use of empirical relations for energy dissipation and provide comparable results with those of extended Boussinesq models (Nwogu, 1993; Madsen et al., 1997; Chen et al., 2000; and Lynett et al., 2002). Such a non-hydrostatic approach, if builds on existing nonlinear shallow-water models with explicit schemes (e.g., Shuto and Goto, 1978; Mader and Curtis, 1991; Kowalik and Murty, 1993; Liu et al., 1995b; Imamura, 1996; and Titov and Synolakis, 1998), will have greater application in the tsunami research community. Numerical stability, however, is a critical issue. The difficulty lies in the flux approximation, in which the velocity and flow depth are evaluated at different locations in a finite difference scheme. The resulting errors in flux estimations often become the source of instability. Mader (1988) proposed a unique upwind scheme that extrapolates the surface elevation instead of the flow depth to determine explicitly the flux in the continuity equation of a nonlinear shallow-water model. Kowalik et al. (2005) implemented this upwind flux approximation in their nonlinear shallow-water model and showed remarkable stability in simulating propagation and runup of the 2004 Indian Ocean Tsunami on a global scale.

This dissertation integrates the experience from previous works and presents a depth-integrated non-hydrostatic model capable of handling dispersion as well as flow discontinuities associated with breaking waves and hydraulic jumps for tsunami modeling. The formulation, which builds on an explicit scheme of the nonlinear shallow-water equations, allows a direct implementation of the upwind flux approximation of Mader (1988) to improve model stability for discontinuous flows. The first-order partial
differential equations facilitate the implementation of two-way nested computational grids for simultaneous computation of basin-wide evolution and coastal runup of tsunamis. The vertical velocity term associated with the non-hydrostatic pressure also describes tsunami generation due to seafloor deformation. Recent advances in seismic inverse algorithms such as Ji et al. (2002) and Honda et al. (2004) allow description of rise time and rupture propagation over the source area, and provide the time series of seafloor displacement as input to the non-hydrostatic model. However, dispersive models based on non-hydrostatic or Boussinesq-type formulations are prone to instabilities created by localized, steep bottom gradients (Horrillo et al., 2006; and Løvholt and Perdersen, 2008). A depth-dependent Gaussian function is implemented to allow smoothing of bathymetric features smaller than the water depth to improve convergence of the implicit, non-hydrostatic solution.

In this dissertation, Chapter 2 describes the derivation of the depth-integrated non-hydrostatic equations from the three-dimensional Navier-Stokes equations and the continuity equation in the spherical coordinate system. Linearization of the governing equations allows derivation of the dispersion relation for comparison with the exact linear dispersion relation from Airy wave theory. Chapter 3 introduces the discretization scheme and provides the numerical formulation for the depth-integrated, non-hydrostatic equations in both spherical and Cartesian grids. Chapter 4 describes auxiliary, but important numerical schemes for implementation of the depth-integrated, non-hydrostatic model in real-field tsunami modeling. These include a wet-dry moving boundary condition, a two-way grid-nesting scheme, and a depth-dependent bathymetry-smoothing strategy. In Chapter 5, simulations of solitary wave propagation in a channel, sinusoidal wave transformation over a submerged bar, and $N$-wave transformation and runup in the Monai Valley experiment provide validations of the dispersion characteristics and the
grid nesting scheme. Chapter 6 focuses on the validation of the shock-capturing capability through modeling of solitary wave transformation and runup over a plane beach and a conical island. These nearshore processes including energetic breaking waves have been used extensively for validation of nonlinear shallow-water and Boussinesq-type models, but examinations of depth-integrated non-hydrostatic models in describing these processes are less immediately evident. Chapter 7 describes a practical application of the present model to reconstruct the 2009 Samoa Tsunami. The advanced source mechanisms and well-recorded water-level and runup data allow validation of the proposed model and method for real-field events. Chapter 8 provides the conclusions of this study and describes future research directions.
CHAPTER 2
THEORETICAL FORMULATION

This section re-derives the governing equations for non-hydrostatic free surface flows in the spherical coordinate system from Casulli (1995) and Stelling and Zijlema (2003). The formulation starts with the three-dimensional governing equations and their depth integration together with a linear approximation of the non-hydrostatic pressure for weakly dispersive waves. The explicit use of the vertical velocity allows extension of the original formulation to include seafloor displacement for modeling of tsunami generation from earthquake rupture. The linearized, depth-integrated governing equations provide a theoretical assessment of the dispersion characteristics of the non-hydrostatic formulation for the first time.

2.1 Three-dimensional Governing Equations

Figure 2.1 provides a schematic of the free-surface flow generated by seafloor deformation. The governing equations for the depth-integrated, non-hydrostatic flow are

![Figure 2.1 Schematic of the free-surface flow generated by seafloor deformation.](image)
derived in the spherical coordinate system from the three-dimensional Navier-Stokes equations and the continuity equation. Gill (1982) and Kowalik and Murty (1993) describe the governing equations for three-dimensional basin-wide ocean circulations in the spherical coordinates \((\lambda, \phi, z)\), in which \(\lambda\) is longitude, \(\phi\) is latitude, and \(z\) denotes distance normal to the earth surface, in the form:

\[
\frac{\partial u}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{v}{R \partial \phi} + w \frac{\partial u}{\partial z} = \left(2\Omega + \frac{u}{R \cos \phi}\right)(v \sin \phi - w \cos \phi)
\]

\[
= -\frac{1}{\rho R \cos \phi} \frac{\partial p}{\partial \lambda} + A_x \tag{2.1}
\]

\[
\frac{\partial v}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial v}{\partial \lambda} + \frac{v}{R \partial \phi} + w \frac{\partial v}{\partial z} + \frac{w v}{R} + \left(2\Omega + \frac{u}{R \cos \phi}\right)u \sin \phi = -\frac{1}{\rho R \cos \phi} \frac{\partial p}{\partial \phi} + A_y \tag{2.2}
\]

\[
\frac{\partial w}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial w}{\partial \lambda} + \frac{v}{R \partial \phi} + w \frac{\partial w}{\partial z} - \frac{v^2}{R} - \left(2\Omega + \frac{u}{R \cos \phi}\right)u \cos \phi = -\frac{1}{\rho} \frac{\partial p}{\partial z} + A_z - g \tag{2.3}
\]

\[
\frac{1}{R \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{1}{R \cos \phi} \frac{\partial (v \cos \phi)}{\partial \phi} + \frac{\partial w}{\partial z} = 0 \tag{2.4}
\]

where \(R\) is the earth’s radius; \((u, v, w)\) is flow velocity; \(t\) is time; \(\Omega\) is the earth’s angular velocity; \(\rho\) is water density; \(p\) is pressure; \(g\) is gravitational acceleration; and \((A_x, A_y, A_z)\) is the viscous force. The three-dimensional governing equations fully describe important physics associated with tsunami modeling, but can be computationally intensive for practical applications.

Following Casulli (1995), the pressure is decomposed into hydrostatic and non-hydrostatic components as

\[
p = \rho g (\zeta - z) + p_q \tag{2.5}
\]

where \(p_q\) denotes the non-hydrostatic pressure. The \(z\) derivative of the hydrostatic pressure term in (2.3) becomes
\[- \frac{1}{\rho} \frac{\partial \left[ \rho g (\zeta - z) \right]}{\partial z} = - \frac{\rho g}{\rho} (0 - 1) = g \]  
\hspace{5cm} (2.6)

With the pressure decomposition, the Navier-Stokes equations (2.1)-(2.3) reduce to the three-dimensional non-hydrostatic momentum equations as

\[
\frac{\partial u}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{v}{R \phi} \frac{\partial u}{\partial \phi} + w \frac{\partial u}{\partial z} - \left( 2\Omega + \frac{u}{R \cos \phi} \right) \left( v \sin \phi - w \cos \phi \right) = - \frac{1}{R \cos \phi} \left( \frac{\partial \zeta}{\partial \lambda} + \frac{\partial q}{\partial \lambda} \right) + A_{\lambda} \hspace{2cm} (2.7)
\]

\[
\frac{\partial v}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial v}{\partial \lambda} + \frac{v}{R \phi} \frac{\partial v}{\partial \phi} + w \frac{\partial v}{\partial z} + \frac{w v}{R} + \left( 2\Omega + \frac{u}{R \cos \phi} \right) u \sin \phi = - \frac{1}{R} \left( \frac{\partial \zeta}{\partial \phi} + \frac{\partial q}{\partial \phi} \right) + A_{\phi} \hspace{2cm} (2.8)
\]

\[
\frac{\partial w}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial w}{\partial \lambda} + \frac{v}{R \phi} \frac{\partial w}{\partial \phi} + w \frac{\partial w}{\partial z} - \frac{v^2}{R} - \left( 2\Omega + \frac{u}{R \cos \phi} \right) u \cos \phi = - \frac{\partial q}{\partial z} + A_{z} \hspace{2cm} (2.9)
\]

Note the pressure in the vertical momentum equation (2.9) is driven primarily by the non-hydrostatic components through the vertical velocity. The three-dimensional non-hydrostatic formulation based on (2.7)-(2.9) and (2.4) has been extensively applied to modeling of free surface flows involving ocean and coastal circulations (e.g., Mahadevan et al., 1996a; Casulli and Stelling, 1998; Stansby and Zhou, 1998; and Stelling and Busnelli, 2001). These models first compute the more stable hydrostatic solution and then correct the solution with the non-hydrostatic pressure, which is implicitly solved from a Poisson equation derived from the continuity equation. Mahadevan et al. (1996b) shows that the pressure decomposition (2.5) in the non-hydrostatic formulation provides a more stable solution and significantly improves computational efficiency in comparison to a direct solution of (2.1)-(2.4).
Even though the pressure decomposition (2.5) improves computational efficiency, the three-dimensional non-hydrostatic models are still computationally expensive. These models require a large number of vertical grid cells or layers, restricting the application to a small area or the computation with coarse resolution. It is practically impossible to model tsunamis from generation, propagation, to runup with appropriate resolution using a three-dimensional, non-hydrostatic model.

2.2 Depth-integrated Governing Equations

The earlier three-dimensional models define the non-hydrostatic pressure at the cell-center, which requires a large number of vertical grid cells to describe its variation. Stelling and Zijlema (2003) proposed a non-hydrostatic model with multi-layer and depth-integrated formulations, which take advantage of the Keller box scheme (1971) for vertical gradient approximation. The Keller box scheme (1971) defines the non-hydrostatic pressure at cell interfaces to estimate the vertical gradient, and with a few vertical layers, provides equivalent solutions to those of the three-dimensional non-hydrostatic and Navier-Stokes models. The non-hydrostatic model can be developed in a depth-integrated formulation by applying the Keller box scheme as in the multi-layer formulation. A depth-integrated non-hydrostatic model is much more efficient and enables modeling of the tsunami evolution processes with desired grid resolution over an extended region.

The depth-integrated formulation is derived from the three dimensional, non-hydrostatic governing equations (2.7)-(2.9) and (2.4). The boundary conditions at the free surface and seafloor facilitate the formulation of the depth-integrated governing equations through the Keller box scheme (1971). The flow depth is the distance between the two boundaries defined as
\[ D = \zeta + (h - \eta) \]  

(2.10)

where \( \zeta \) is the surface elevation measured from the still-water level, \( h \) is the water depth defined at the still water level, and \( \eta \) is the seafloor displacement. Material derivatives of the surface elevation and water depth give rise to the kinematic free surface and seafloor boundary conditions as

\[
w = \frac{\partial \zeta}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial \zeta}{\partial \lambda} + \frac{v}{R} \frac{\partial \zeta}{\partial \phi} \quad \text{at } z = \zeta
\]

(2.11)

\[
w = \frac{\partial \eta}{\partial t} - \frac{u}{R \cos \phi} \frac{\partial (h - \eta)}{\partial \lambda} - \frac{v}{R} \frac{\partial (h - \eta)}{\partial \phi} \quad \text{at } z = -h + \eta
\]

(2.12)

Both the hydrostatic and non-hydrostatic pressure terms vanish at \( z = \zeta \) to provide the dynamic free surface boundary condition.

Tsunamis are long waves with weak dispersion characteristics. The vertical velocity \( w \) generally follows a linear distribution between the seafloor and the free surface. The Coriolis effects, the viscous dissipation, and the variation of the horizontal velocity in the vertical direction are negligible in tsunami propagation. The kinematic boundary conditions (2.11) and (2.12) are simplified to

\[
w = \frac{\partial \zeta}{\partial t} + \frac{U}{R \cos \phi} \frac{\partial \zeta}{\partial \lambda} + \frac{V}{R} \frac{\partial \zeta}{\partial \phi} \quad \text{at } z = \zeta
\]

(2.13)

\[
w = \frac{\partial \eta}{\partial t} - \frac{U}{R \cos \phi} \frac{\partial (h - \eta)}{\partial \lambda} - \frac{V}{R} \frac{\partial (h - \eta)}{\partial \phi} \quad \text{at } z = -h + \eta
\]

(2.14)

where \( U \) and \( V \) are depth-averaged velocity components in the \( \lambda \) and \( \phi \) directions, and the vertical momentum equation (2.9) reduces to

\[
\frac{\partial w}{\partial t} = -\frac{\partial q}{\partial z}
\]

(2.15)
Depth integration of (2.7), (2.8), (2.15), and (2.4) taking into account of the kinematic boundary conditions (2.13) and (2.14) provides

\[ \frac{\partial U}{\partial t} + \frac{U}{R \cos \phi} \frac{\partial U}{\partial \lambda} + \frac{V}{R \phi} \frac{\partial U}{\partial \phi} = \left(2\Omega + \frac{U}{R \cos \phi}\right) V \sin \phi = -\frac{g}{R \cos \phi} \frac{\partial \zeta}{\partial \phi} - \frac{1}{DR \cos \phi} \int_{-k+\eta}^{\zeta} \frac{\partial q}{\partial \phi} dz - f \frac{U \sqrt{U^2 + V^2}}{D} \]  

(2.16)

\[ \frac{\partial V}{\partial t} + \frac{U}{R \cos \phi} \frac{\partial V}{\partial \lambda} + \frac{V}{R \phi} \frac{\partial V}{\partial \phi} = \left(2\Omega + \frac{U}{R \cos \phi}\right) U \sin \phi = -\frac{g}{R \cos \phi} \frac{\partial \zeta}{\partial \phi} - \frac{1}{DR \cos \phi} \int_{-k+\eta}^{\zeta} \frac{\partial q}{\partial \phi} dz - f \frac{V \sqrt{U^2 + V^2}}{D} \]  

(2.17)

\[ \frac{\partial W}{\partial t} = -\frac{1}{D} \int_{-k+\eta}^{\zeta} \frac{\partial q}{\partial z} dz \]  

(2.18)

\[ \frac{\partial (\zeta - \eta)}{\partial t} + \frac{1}{R \cos \phi} \frac{\partial (UD)}{\partial \lambda} + \frac{1}{R \cos \phi} \frac{\partial (V \cos \phi D)}{\partial \phi} = 0 \]  

(2.19)

where \( W \) is the depth-averaged velocity component in the \( z \) direction; \( f \) is a dimensionless friction factor given in terms of Manning’s relative roughness coefficient \( n \) as

\[ f = n^2 \frac{g}{D^{1/3}} \]  

(2.20)

Standard hydraulic texts such as Chow (1959) and Chaudhry (1993) provide the Manning coefficient as a function of materials and surface conditions, while Bretschneider et al. (1986) determined typical values of the coefficient for tropical island terrain and vegetation through measurements of wind profiles.

The momentum equations (2.16)-(2.18) contain integrals of the non-hydrostatic pressure and the way they are evaluated will influence the dispersion characteristics of the
resulting governing equations. The non-hydrostatic pressure term in (2.16) can be derived through the Leibniz rule of integration:

\[
\frac{\partial}{\partial \lambda} \int_{-h+\eta}^{\zeta} q'dz = \int_{-h+\eta}^{\zeta} \frac{\partial q}{\partial \lambda} dz + q_{\zeta} \frac{\partial}{\partial \lambda} - q_{-h+\eta} \frac{\partial(-h+\eta)}{\partial \lambda}
\] (2.21)

where \(q_{\zeta}\) and \(q_{-h+\eta}\) are non-hydrostatic pressure at the free surface and seafloor, respectively. The integration of non-hydrostatic pressure on the left hand side of (2.21) is approximated as

\[
\frac{\partial}{\partial \lambda} \int_{-h+\eta}^{\zeta} q'dz \approx \frac{\partial}{\partial \lambda} \left[D \left( \frac{q_{\zeta} + q_{-h+\eta}}{2} \right) \right]
\] (2.22)

Using (2.21) and (2.22), the non-hydrostatic pressure term in (2.16) can be rewritten in the form:

\[
\int_{-h+\eta}^{\zeta} \frac{\partial q}{\partial \lambda} dz = \frac{1}{2} \frac{\partial q}{\partial \lambda} - q_{-h+\eta} \frac{\partial(-h+\eta)}{\partial \lambda}
\] (2.23)

The same procedure gives the non-hydrostatic pressure term in the \(\phi\)-momentum equation (2.17) as

\[
\int_{-h+\eta}^{\zeta} \frac{\partial q}{\partial \phi} dz = \frac{1}{2} \frac{\partial q}{\partial \phi} - q_{-h+\eta} \frac{\partial(-h+\eta)}{\partial \phi}
\] (2.24)

Note that \(q_{\zeta} = 0\) since the total pressure vanishes at the free surface. The vertical momentum equation (2.18) can be simplified as

\[
\frac{\partial W}{\partial t} = \frac{q_{-h+\eta}}{D}
\] (2.25)

The non-hydrostatic pressure in (2.23), (2.24), and (2.25) are now expressed in terms of \(q_{-h+\eta}\) defined on the seafloor.

Substitution of the non-hydrostatic pressure terms (2.23) and (2.24) into the horizontal momentum equations (2.16) and (2.17) and the simplification of the z momentum
equation from (2.25), along with the continuity equation (2.19), give rise to the depth-integrated, non-hydrostatic governing equations:

\[
\frac{\partial U}{\partial t} + \frac{U}{R \cos \phi} \frac{\partial U}{\partial \lambda} + \frac{V}{R \cos \phi} \frac{\partial U}{\partial \phi} - \left( 2\Omega + \frac{U}{R \cos \phi} \right) V \sin \phi = -\frac{g}{R \cos \phi} \frac{\partial \zeta}{\partial \lambda} - \frac{1}{2} \frac{\partial q}{\partial \lambda} - \frac{1}{2} R \frac{\partial (\zeta - h + \eta)}{\partial \lambda} - f \frac{U \sqrt{U^2 + V^2}}{D} \tag{2.26}
\]

\[
\frac{\partial V}{\partial t} + \frac{U}{R \cos \phi} \frac{\partial V}{\partial \lambda} + \frac{V}{R \cos \phi} \frac{\partial V}{\partial \phi} + \left( 2\Omega + \frac{U}{R \cos \phi} \right) U \sin \phi = -\frac{g}{R \cos \phi} \frac{\partial \zeta}{\partial \phi} - \frac{1}{2} \frac{\partial q}{\partial \phi} - \frac{1}{2} R \frac{\partial (\zeta - h + \eta)}{\partial \phi} - f \frac{V \sqrt{U^2 + V^2}}{D} \tag{2.27}
\]

\[
\frac{\partial W}{\partial t} = \frac{q}{D} \tag{2.28}
\]

\[
\frac{\partial (\zeta - \eta)}{\partial t} + \frac{1}{R \cos \phi} \frac{\partial (U D)}{\partial \lambda} + \frac{1}{R \cos \phi} \frac{\partial (V \cos \phi D)}{\partial \phi} = 0 \tag{2.29}
\]

where \( q \) is now defined as the non-hydrostatic pressure at the seafloor. Because of the assumption of a linear distribution for \( w \), the vertical velocity is simply

\[
W = \frac{w_\zeta + w_{-\zeta + \eta}}{2} \tag{2.30}
\]

which is the average value of \( w \) at the free surface and the seafloor given respectively in (2.13) and (2.14). Except for the addition of the vertical momentum equation and the non-hydrostatic pressure terms in the horizontal momentum equations, the governing equations have the same structure as the nonlinear shallow-water equations. This formulation allows a straightforward extension of existing nonlinear shallow-water models for non-hydrostatic flows.
The governing equations in the spherical coordinate system \((\lambda, \phi)\) can be transformed into the Cartesian system \((x, y)\) for a relatively small region, where effects of the earth’s curvature is insignificant. The distance in the \(x\) and \(y\) directions becomes

\[ x = R\lambda \cos \phi, \quad y = R\phi \]  \hspace{1cm} (2.31)

The Coriolis terms in the horizontal momentum equations (2.26) and (2.27) vanish

\[
\left( 2\Omega + \frac{U}{R \cos \phi} \right) V \sin \phi = 0, \quad \left( 2\Omega + \frac{U}{R \cos \phi} \right) U \sin \phi = 0
\]  \hspace{1cm} (2.32)

Since the scale does not taper off in the north-south direction, we can simply set \(\phi = 0\).

These modifications provide the governing equations in the Cartesian coordinate system in the form:

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -g \frac{\partial \zeta}{\partial x} - \frac{1}{2} q \frac{\partial}{\partial x} \left( \frac{\partial (\zeta - h + \eta)}{\partial x} \right) - f \frac{U \sqrt{U^2 + V^2}}{D}
\]  \hspace{1cm} (2.33)

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -g \frac{\partial \zeta}{\partial y} - \frac{1}{2} q \frac{\partial}{\partial y} \left( \frac{\partial (\zeta - h + \eta)}{\partial y} \right) - f \frac{V \sqrt{U^2 + V^2}}{D}
\]  \hspace{1cm} (2.34)

\[
\frac{\partial W}{\partial t} = \frac{q}{D}
\]  \hspace{1cm} (2.35)

\[
\frac{\partial (\zeta - \eta)}{\partial t} + \frac{\partial (UD)}{\partial x} + \frac{\partial (VD)}{\partial y} = 0
\]  \hspace{1cm} (2.36)

where \(U\) and \(V\) are now the depth-averaged velocity components in the \(x\) and \(y\) directions.

The non-hydrostatic pressure is expressed in terms of the vertical acceleration through the momentum equation (2.35). The kinematic boundary conditions (2.13) and (2.14) describe the average vertical velocity in (2.35) as functions of \(U, V, \zeta,\) and \(\eta\) to close the depth-integrated governing equations for non-hydrostatic flows.

The advantage of the depth-integrated non-hydrostatic formulation is the form of the dispersive terms, which are only the first derivative of the non-hydrostatic pressure. This
can be achieved because the non-hydrostatic pressure directly relates to the vertical acceleration. On the other hand, the dispersive terms in Boussinesq-type equations are expressed in term of the horizontal velocity, which requires additional temporal and spatial differentiation to relate to the non-hydrostatic pressure.

2.3 Linear Dispersion Relation

The depth-integrated non-hydrostatic equations describe wave dispersion through the non-hydrostatic pressure and vertical velocity. Although the basic assumptions are consistent with those in the classical Boussinesq equations of Peregrine (1967), the dispersion characteristics might differ due to the variables employed and truncation of terms in the derivation of the governing equations. This section derives the linear dispersion relation for the depth-integrated non-hydrostatic equations for direct comparison with those from the classical Boussinesq equations and Airy wave theory. Although the non-hydrostatic and the Boussinesq equations are nonlinear, the comparison of their linear dispersion characteristics provides a baseline for assessment.

The first step is to derive a linearized version of the depth-integrated, non-hydrostatic governing equations. After dropping the nonlinear terms and setting $\eta = 0$, the governing equations in the $x$ direction from (2.33) to (2.36) become

\begin{align}
\frac{\partial U}{\partial t} + g \frac{\partial \zeta}{\partial x} + \frac{1}{2} \frac{\partial q}{\partial x} - \frac{1}{2} g \frac{\partial h}{\partial x} &= 0 \quad (2.37) \\
\frac{\partial}{\partial t} \left( \frac{w_z + w_{-h}}{2} \right) - \frac{q}{h} &= 0 \quad (2.38) \\
\frac{\partial \zeta}{\partial t} + h \frac{\partial U}{\partial x} &= 0 \quad (2.39)
\end{align}
The one-dimensional kinematic conditions on the free surface and the seafloor, with the vertical variation of the horizontal velocity neglected and \( \eta = 0 \), become

\[
\begin{align*}
\frac{w_z}{c} &= \frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} \\
\frac{w_h}{c} &= -U \frac{\partial h}{\partial x}
\end{align*}
\] (2.40)

The non-hydrostatic pressure \( q \) in terms of \( \zeta \) and \( U \) can be obtained from the vertical momentum equation (2.38) by substitution of the kinematic boundary conditions (2.40) and (2.41) as

\[
q = \frac{1}{2} h \frac{\partial^2 \zeta}{\partial t^2} + \frac{1}{2} h U \frac{\partial^2 \zeta}{\partial t \partial x} + \frac{1}{2} \frac{\partial U}{\partial t} \frac{\partial (\zeta - h)}{\partial x}
\] (2.42)

The \( x \)-derivative of \( q \) is expanded by the chain rule as

\[
\frac{\partial q}{\partial x} = \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial^2 \zeta}{\partial t^2} + \frac{1}{2} h \frac{\partial^3 \zeta}{\partial t^3} + \frac{1}{2} \frac{\partial U}{\partial x} \frac{\partial^2 \zeta}{\partial t \partial x} + \frac{1}{2} \frac{\partial h}{\partial t} \frac{\partial^2 \zeta}{\partial t \partial x} + \frac{1}{2} \frac{\partial U}{\partial t} \frac{\partial^3 \zeta}{\partial t^3} \frac{\partial (\zeta - h)}{\partial x} + \frac{1}{2} \frac{\partial h}{\partial t} \frac{\partial^2 (\zeta - h)}{\partial t \partial x} + \frac{1}{2} \frac{\partial U}{\partial t} \frac{\partial^2 (\zeta - h)}{\partial x \partial t} (2.43)
\]

Substituting (2.42) and (2.43) into the horizontal momentum equation (2.37), we have the horizontal momentum equation in terms of \( \zeta \) and \( U \) as

\[
\frac{\partial U}{\partial t} + g \frac{\partial \zeta}{\partial x} + \frac{1}{4} h \frac{\partial^3 \zeta}{\partial t \partial \xi} - \frac{1}{4} h \frac{\partial h}{\partial \xi} \frac{\partial^2 U}{\partial \xi \partial t} - \frac{1}{4} \frac{\partial^3 h}{\partial \xi^3} \frac{\partial U}{\partial \xi} - \frac{1}{4} \frac{\partial h}{\partial \xi} \frac{\partial^3 \zeta}{\partial \xi^3} = 0
\] (2.44)

The non-hydrostatic pressure and vertical velocity are now expressed by a third-order derivative of the surface elevation and three additional terms related to the depth gradient.

The exact linear dispersion relation from Airy wave theory is for constant water depth.

The Boussinesq-type equations were initially derived for constant water depth (Peregrine,
1967; and Madsen et al., 1991). To provide a direct comparison, the depth-gradient terms in (2.44) are dropped. The resulting governing equations for constant water depth become

\[ \frac{\partial U}{\partial t} + g \frac{\partial \zeta}{\partial x} + \frac{1}{4} h \frac{\partial^3 \zeta}{\partial t^2 \partial x} = 0 \]  

(2.45)

\[ \frac{\partial \zeta}{\partial t} + h \frac{\partial U}{\partial x} = 0 \]  

(2.46)

Following Madsen et al. (1991) and Nwogu (1993), the linear dispersive relation is derived by considering a system small amplitude periodic waves in the form:

\[ \zeta = \zeta_0 e^{i(k(x-ct))} \]  

(2.47)

\[ U = U_0 e^{i(k(x-ct))} \]  

(2.48)

where \( k \) and \( c \) denote the wave number and celerity. Substituting (2.47) and (2.48) into the linearized depth-integrated, non-hydrostatic equation (2.45) and (2.46) yields the dispersion relation:

\[ c^2 = \frac{gh}{1 + \frac{1}{4} (kh)^2} \]  

(2.49)

The linear dispersion relation from the classical Boussinesq equations of Peregrine (1967) is given by

\[ c^2 = \frac{gh}{1 + \frac{1}{3} (kh)^2} \]  

(2.50)

The exact linear dispersion relation of Airy wave theory is in the form:

\[ c^2 = \frac{gh \tanh(kh)}{kh} \]  

(2.51)
Figure 2.2 Linear dispersion relation. - - - - , exact solution; -- -- --, classical Boussinesq equation of Peregrine (1967); — (red), depth-integrated, non-hydrostatic equations.

Figure 2.2 compares the linear dispersion relations (2.49) and (2.50) with the exact relation (2.51). The applicable range of a model is up to an error of 5% in the linear dispersion relation according to Madsen et al. (1991). Within the intermediate water depth $\pi/10 < kh < \pi$, the dispersion relation of the depth-integrated non-hydrostatic equations has an error less than 5%. The classical Boussinesq equation of Peregrine (1967) has an error of 5% at $kh = 1.35$ and a maximum of 20% within the intermediate depth range. This provides a theoretical proof of the observations by Stelling and Zijlema (2003) and Walters (2005) that their non-hydrostatic models produce better dispersion characteristics than the classical Boussinesq equations.

To gain understanding of the dispersion characteristics between the non-hydrostatic and the Boussinesq equations, the momentum equation (2.45) is rewritten with the continuity equation (2.46) to express the dispersive term as a function of the horizontal velocity. After dropping the bottom-gradient term, we have
\[
\frac{\partial U}{\partial t} + g \frac{\partial \zeta}{\partial x} - \frac{1}{4} h^2 \frac{\partial}{\partial x} \left( \frac{\partial^2 U}{\partial x^2} \right) = 0
\]  

(2.52)

This has the same form as the momentum equation of the linearized classical Boussinesq equations as

\[
\frac{\partial U}{\partial t} + g \frac{\partial \zeta}{\partial x} - \frac{1}{3} h^2 \frac{\partial}{\partial x} \left( \frac{\partial^2 U}{\partial x^2} \right) = 0
\]  

(2.53)

The only difference is in the coefficient of the linear dispersive term that mirrors the respective linear dispersion relations (2.49) and (2.50). The major difference between the two approaches is the use of the irrotational flow condition in the classical Boussinesq equations to express the vertical velocity in terms of the horizontal velocity. This introduces an additional depth-integration step in the derivation of the classical Boussinesq equations. The depth-integrated, non-hydrostatic formulation, on the other hands, utilizes an approximated vertical momentum equation to account for vertical velocity effects, which are weakly coupled with the horizontal velocity through the governing equations. When the vertical velocity variation over a water column deviates from the long-wave assumptions, the additional integration in the classical Boussinesq equations could amplify the error through the horizontal velocity, which is the primary variable in the physical problem. This is reflected in the comparison of the dispersion relations in Figure 2.2.
CHAPTER 3
NUMERICAL FORMULATION

The numerical formulation includes the solution schemes for the hydrostatic and non-hydrostatic components of the governing equations in both the spherical and Cartesian grid systems. The finite difference scheme utilizes the upwind flux approximation of Mader (1988) in the continuity equation as well as the calculation of the advective terms in the horizontal momentum equations. Figure 3.1 shows the space-staggered grid for the computation. The model calculates the horizontal velocity components $U$ and $V$ at the cell interface and the free surface elevation $\zeta$, the non-hydrostatic pressure $q$, and the vertical velocity $W$ at the cell center, where the water depth $h$ is defined.

![Figure 3.1 Definition sketch of spatial grid.](image-url)
3.1 Hydrostatic Solution in Spherical Grids

The hydrostatic model utilizes an explicit scheme for the solution. Integration of the continuity equation (2.29) provides an update of the surface elevation at the center of cell $(j, k)$ in terms of the fluxes, $FLX$ and $FLY$, along the longitude and latitude at the cell interfaces as

$$\xi_{j,k}^{m+1} = \xi_{j,k}^m + \left(\eta_{j,k}^{m+1} - \eta_{j,k}^m\right) - \Delta t \frac{FLX_{j+1,k} - FLX_{j,k}}{R \Delta \lambda \cos \phi_k}$$

$$- \Delta t \frac{FLY_{j,k} \cos(\phi_k + \Delta \phi/2) - FLY_{j,k-1} \cos(\phi_{k-1} + \Delta \phi/2)}{R \Delta \phi \cos \phi_k} \tag{3.1}$$

where $m$ denotes the time step, $\Delta t$ the time step size, and $\Delta \lambda$ and $\Delta \phi$ the respective grid sizes. The upwind scheme gives the flux terms as

$$FLX_{j,k} = U_p^m \xi_{j-1,k}^m + U_n^m \xi_{j,k}^m + U_{j,k}^{m+1} \frac{(h_{j-1,k} - \eta_{j,k}^m) + (h_{j,k} - \eta_{j,k}^m)}{2} \tag{3.2a}$$

$$FLY_{j,k} = V_p^m \xi_{j,k}^m + V_n^m \xi_{j,k+1}^m + V_{j,k}^{m+1} \frac{(h_{j,k} - \eta_{j,k}^m) + (h_{j,k+1} - \eta_{j,k+1}^m)}{2} \tag{3.2b}$$

in which

$$U_p^m = \frac{U_p^m}{2}, \quad U_n^m = \frac{U_n^m}{2}, \quad V_p^m = \frac{V_p^m}{2}, \quad V_n^m = \frac{V_n^m}{2} \tag{3.3}$$

The upwind flux approximation (3.2) extrapolates the surface elevation from the upwind cell, while the water depth takes on the average value from the two adjacent cells (Mader, 1988). This represents a departure from most existing shallow-water models that extrapolate the flow depth instead (Kowalik and Murty, 1993; and Titov and Synolakis, 1998). In these models, the flux is determined with a second-order scheme

$$FLX_{j,k} = U_{j,k}^{m+1} \frac{(\xi_{j-1,k}^{m+1} + h_{j-1,k} - \eta_{j-1,k}^m) + (\xi_{j,k}^m + h_{j,k} - \eta_{j,k}^m)}{2} \tag{3.4a}$$
This approach uses average values of the surface elevations and water depths from adjacent cells to determine the flux and is equivalent to the flux-based formulation of the nonlinear shallow-water equations (e.g., Shuto and Goto, 1978; Liu et al., 1995b; and Imamura, 1996).

The horizontal momentum equations provide the velocity components $U$ and $V$ at $(m+1)$ in (3.2) for the update of the surface elevation in (3.1). In the spatial discretization, the average values of $U$ and $V$ are used in the horizontal momentum equations. These average velocity components are defined by

$$
\bar{U}^m_{j,k} = \frac{1}{4} \left( U^m_{j,k} + U^m_{j+1,k} + U^m_{j+1,k+1} + U^m_{j,k+1} \right)
$$

$$
\bar{V}^m_{x,j,k} = \frac{1}{4} \left( V^m_{j,k} + V^m_{j-1,k} + V^m_{j-1,k-1} + V^m_{j,k-1} \right)
$$

Integration of the $\lambda$ and $\phi$-momentum equations (2.26) and (2.27), with the non-hydrostatic terms omitted, provides the hydrostatic solution for the horizontal velocity

$$
\vec{U}^{m+1}_{j,k} = U^m_{j,k} - \frac{g \Delta t}{R \Delta \lambda \cos \phi_k} (\zeta^m_{j,k} - \zeta^m_{j-1,k}) + \Delta t \left( 2 \Omega + \frac{U^m_{j,k}}{R \cos \phi_k} \right) \bar{V}^m_{x,j,k} \sin \phi_k
$$

$$
- \frac{\Delta t}{R \Delta \phi} \bar{V}^m_{x,p} (U^m_{j,k} - U^m_{j,k-1}) - \frac{\Delta t}{R \Delta \phi} \bar{V}^m_{x,n} (U^m_{j,k+1} - U^m_{j,k})
$$

$$
- \frac{\Delta t}{R} \bar{V}^m_{x,p} (U^m_{j,k} - U^m_{j,k-1}) - \frac{\Delta t}{R \Delta \phi} \bar{V}^m_{x,n} (U^m_{j,k+1} - U^m_{j,k})
$$

$$
- n^2 g \frac{\Delta t U^m_{j,k} \sqrt{(U^m_{j,k})^2 + (\bar{V}^m_{x,j,k})^2}}{(D^m_{j-1,k} + D^m_{j,k})^{\frac{3}{2}}}
$$

(3.7)
\[
\tilde{V}_{j,k}^{m+1} = V_{j,k}^m - \frac{2\Delta t}{R\Delta \phi} \left( \zeta_{j,k+1}^m - \zeta_{j,k}^m \right) - \frac{\Delta t}{R\Delta \phi \cos(\phi_k + \Delta \phi/2)} \left( 2\Omega + \frac{\bar{U}_{y,j,k}^m}{R \cos(\phi_k + \Delta \phi/2)} \right) \bar{U}_{y,j,k}^m \sin(\phi_k + \Delta \phi/2)
\]

\[
- \frac{\Delta t}{R\Delta \phi \cos(\phi_k + \Delta \phi/2)} \bar{U}_{y,j,k}^m (V_{j,k}^m - V_{j-1,k}^m) - \frac{\Delta t}{R\Delta \phi} V_{j,k}^m (V_{j,k+1}^m - V_{j,k}^m)
\]

\[
- n^2 g \left( \frac{\Delta t V_{j,k}^m \left( \bar{U}_{y,j,k}^m \right)^2 + \left( V_{j,k}^m \right)^2}{\left( D_{j,k}^m + D_{j,k+1}^m \right)^{4/3}} \right)
\]

(3.8)

where the subscripts \( p \) and \( n \) indicates upwind and downwind approximations of the advective speeds.

Most nonlinear shallow-water models, which use the advective speeds from (3.3) in the momentum equations, cannot capture flow discontinuities associated with breaking waves or hydraulic jumps. Stelling and Duinmeijer (2003) derived an alternative discretization of the advective speeds that conserves energy or momentum across flow discontinuities. Momentum conservation provides a better description of bores or hydraulic jumps that mimic breaking waves in depth-integrated flows. This is also consistent with the finite volume method with a Riemann solver (e.g., Wei et al., 2006; Wu and Cheung, 2008; and George, 2010). We adapt the momentum-conserved advection scheme from Stelling and Duinmeijer (2003) with the present upwind flux approach to provide the advective speeds as

\[
U_p^m = \begin{cases} 
\hat{U}_{p,j,k}^m + \hat{U}_{p,j,k}^m \over 2 & \text{if } U_{j,k}^m \neq 0 \\
0 & \text{if } U_{j,k}^m = 0 
\end{cases}, \quad U_n^m = \begin{cases} 
\hat{U}_{n,j,k}^m - \hat{U}_{n,j,k}^m \over 2 & \text{if } U_{j,k}^m \neq 0 \\
0 & \text{if } U_{j,k}^m = 0 
\end{cases}
\]

(3.9a)
\[
V_p^m = \begin{cases} 
\frac{\hat{V}_{p,j,k}^m + \hat{V}_{p,j,k}^m}{2} & \text{if } V_{j,k}^m \neq 0, \\
0 & \text{if } V_{j,k}^m = 0
\end{cases}, \\
V_n^m = \begin{cases} 
\frac{\hat{V}_{n,j,k}^m - \hat{V}_{n,j,k}^m}{2} & \text{if } V_{j,k}^m \neq 0, \\
0 & \text{if } V_{j,k}^m = 0
\end{cases}
\] (3.9b)

in which

\[
\hat{U}_{p,j,k}^m = \frac{2FLU_{p,j,k}^m}{D_{j-1,k}^m + D_{j,k}^m}, \quad \hat{U}_{n,j,k}^m = \frac{2FLU_{n,j,k}^m}{D_{j-1,k}^m + D_{j,k}^m}
\] (3.10a)

\[
\hat{V}_{p,j,k}^m = \frac{2FLV_{p,j,k}^m}{D_{j,k}^m + D_{j,k+1}^m}, \quad \hat{V}_{n,j,k}^m = \frac{2FLV_{n,j,k}^m}{D_{j,k}^m + D_{j,k+1}^m}
\] (3.10b)

where the flux for a positive flow \((U_{j,k}^m > 0)\) is given by

\[
FLU_{p,j,k}^m = \begin{cases} 
\frac{U_{j-1,k}^m + U_{j,k}^m}{2} \left( h_{j-1,k} - \eta_{j-1,k}^m + \frac{\xi_{j-2,k}^m + \xi_{j-1,k}^m}{2} \right) & \text{if } U_{j-1,k}^m > 0, \\
\frac{U_{j-1,k}^m + U_{j,k}^m}{2} \left( h_{j-1,k} - \eta_{j-1,k}^m + \xi_{j-1,k}^m \right) & \text{if } U_{j-1,k}^m < 0 \text{ and } |U_{j,k}^m| \geq |U_{j-1,k}^m|, \\
U_{j,k}^m \frac{D_{j-1,k}^m + D_{j,k}^m}{2} & \text{if } U_{j-1,k}^m < 0 \text{ and } |U_{j,k}^m| < |U_{j-1,k}^m|
\end{cases}
\] (3.11a)

and the flux for a negative flow \((U_{j,k}^m < 0)\) is

\[
FLU_{n,j,k}^m = \begin{cases} 
\frac{U_{j,k}^m + U_{j+1,k}^m}{2} \left( h_{j,k} - \eta_{j,k}^m + \frac{\xi_{j,k}^m + \xi_{j+1,k}^m}{2} \right) & \text{if } U_{j+1,k}^m < 0, \\
\frac{U_{j,k}^m + U_{j+1,k}^m}{2} \left( h_{j,k} - \eta_{j,k}^m + \xi_{j,k}^m \right) & \text{if } U_{j+1,k}^m > 0 \text{ and } |U_{j,k}^m| \geq |U_{j+1,k}^m|, \\
U_{j,k}^m \frac{D_{j-1,k}^m + D_{j,k}^m}{2} & \text{if } U_{j+1,k}^m > 0 \text{ and } |U_{j,k}^m| < |U_{j+1,k}^m|
\end{cases}
\] (3.11b)

Similarly, the flux for \(V_{j,k}^m > 0\) is given by
and the flux for \( V_{j,k} < 0 \) is

\[
\begin{align*}
\text{FLV}_{p,j,k}^m &= \begin{cases} \\
\frac{V_{j,k-1}^m + V_{j,k}^m}{2} \left( h_{j,k} - \eta_{j,k}^m + \frac{\zeta_{j,k-1}^m + \zeta_{j,k}^m}{2} \right) & \text{if } V_{j,k-1}^m < 0 \\
\frac{V_{j,k-1}^m + V_{j,k}^m}{2} \left( h_{j,k} - \eta_{j,k}^m + \zeta_{j,k}^m \right) & \text{if } V_{j,k-1}^m > 0 \text{ and } \left| V_{j,k}^m \right| \geq \left| V_{j,k-1}^m \right| \\
V_{j,k}^m \frac{D_{j,k}^m + D_{j,k+1}^m}{2} & \text{if } V_{j,k-1}^m > 0 \text{ and } \left| V_{j,k}^m \right| < \left| V_{j,k-1}^m \right|
\end{cases}
\end{align*}
\]

(3.11c)

The advective speeds \( \overline{U}_{yp}, \overline{U}_{yn}, \overline{V}_{xp}, \) and \( \overline{V}_{xn} \) in (3.7) and (3.8) can be obtained from (3.9) to (3.11) with average values of \( U_{j,k} \) and \( V_{j,k} \) in the form of (3.5) and (3.6) as well as the average values of \( \zeta_{j,k} \) and \( h_{j,k} \) calculated in the same way. This completes the numerical formulation for the nonlinear shallow-water equations in describing hydrostatic flows.

In comparison, Stelling and Duinmeijer (2003) derived the momentum-conserved advection approximation from the conservative form of the nonlinear shallow-water equations. The resulting momentum equations are the same as the non-conservative form with the exception of the advective speeds, which are given by

\[
\hat{U}_{p,j,k}^m = \frac{2\text{FLU}_{j-1,k}^m}{D_{j-1,k}^m + D_{j,k}^m}, \quad \hat{U}_{n,j,k}^m = \frac{2\text{FLU}_{j,k}^m}{D_{j-1,k}^m + D_{j,k}^m}
\]

(3.12)

The flux at the cell center is obtained from
where

\[ \text{FLU}^m_{j,k} = \frac{\text{FLU}^m_{j,k} + \text{FLU}^m_{j+1,k}}{2} \] (3.13)

Note that the averaged fluxes \( \overline{\text{FLU}}^m_{j-1,k} \) and \( \overline{\text{FLU}}^m_{j,k} \) in (3.12) are equivalent to \( \text{FLU}^m_{p,j,k} \) and \( \text{FLU}^m_{n,j,k} \) in (3.11a) and (3.11b) of the present scheme. Their upwind scheme (3.14) extrapolates the flow depth and calculates the average flux \( \overline{\text{FLU}}^m_{j,k} \) at the cell center from the interface values through (3.13). In contrast, the present approach extrapolates the surface elevation at the cell interface and directly calculates the fluxes \( \text{FLU}^m_{p,j,k} \) and \( \text{FLU}^m_{n,j,k} \) at the cell center from (3.11a) and (3.11b). With the water depth defined at the cell center, any discontinuity is due entirely to the surface elevation. The present upwind scheme avoids errors from depth extrapolation and most importantly improves the capability to capture flow discontinuities. This becomes essential when the topography is irregular and wave breaking is energetic.

### 3.2 Non-hydrostatic Solution in Spherical Grids

This section describes the development of the non-hydrostatic solution from the nonlinear shallow-water results as well as the bottom pressure and vertical velocity terms neglected in the hydrostatic formulation. Integration of the non-hydrostatic terms in the horizontal momentum equations completes the update of the horizontal velocity from (3.7) and (3.8)

\[ U^{m+1}_{j,k} = \tilde{U}^{m+1}_{j,k} - \frac{\Delta t}{R \Delta \phi \cos \phi_k} A_{j,k} \left( \frac{q^{m+1}_{j,k} + q^{m+1}_{j-1,k}}{2} \right) \left( \frac{d^{m+1}_{j,k} - d^{m+1}_{j-1,k}}{2} \right) \] (3.15)

\[ V^{m+1}_{j,k} = \tilde{V}^{m+1}_{j,k} - \frac{\Delta t}{R \Delta \phi} B_{j,k} \left( \frac{q^{m+1}_{j,k+1} + q^{m+1}_{j,k}}{2} \right) - \frac{\Delta t}{R \Delta \phi} \left( \frac{d^{m+1}_{j,k+1} - d^{m+1}_{j,k}}{2} \right) \] (3.16)
where

\[
A_{j,k} = \frac{\left(\xi_{j,k}^m - h_{j,k} + \eta_{j,k}^m \right) - \left(\xi_{j-1,k}^m - h_{j-1,k} + \eta_{j-1,k}^m \right)}{D_{j,k}^m + D_{j-1,k}^m}
\]  
\[\text{and} \quad (3.17a)\]

\[
B_{j,k} = \frac{\left(\xi_{j,k+1}^m - h_{j,k+1} + \eta_{j,k+1}^m \right) - \left(\xi_{j,k}^m - h_{j,k} + \eta_{j,k}^m \right)}{D_{j,k}^m + D_{j,k+1}^m}
\]  
\[\text{and} \quad (3.17b)\]

Discretization of the vertical momentum equation (2.28) gives the vertical velocity at the free surface as

\[
w_{s,j,k}^{m+1} = w_{s,j,k}^{m+1} - \left(w_{h,j,k}^{m+1} - w_{b,j,k}^{m+1}\right) + \frac{2\Delta t}{D_{j,k}^m} q_{j,k}^{m+1}
\]  
\[\text{and} \quad (3.18)\]

The vertical velocity at the seafloor is evaluated from the boundary condition (2.14) as

\[
w_{b,j,k}^{m+1} = \frac{\eta_{j,k}^{m+1} - \eta_{j,k}^{m}}{\Delta t}
\]

\[
- \frac{\mathcal{U}_{\text{zp}}^m (h_{j,k} - \eta_{j,k}^m) - (h_{j-1,k} - \eta_{j-1,k}^m)}{R \Delta \lambda \cos \phi_k} - \frac{\mathcal{U}_{\text{zn}}^m (h_{j-1,k} - \eta_{j-1,k}^m) - (h_{j,k} - \eta_{j,k}^m)}{R \Delta \lambda \cos \phi_k}
\]

\[
- \frac{\mathcal{V}_{\text{zp}}^m (h_{j,k} - \eta_{j,k}^m) - (h_{j,k-1} - \eta_{j,k-1}^m)}{R \Delta \phi} - \frac{\mathcal{V}_{\text{zn}}^m (h_{j,k-1} - \eta_{j,k-1}^m) - (h_{j,k} - \eta_{j,k}^m)}{R \Delta \phi}
\]  
\[\text{and} \quad (3.19)\]

in which

\[
\mathcal{U}_{\text{zp}}^m = \frac{U_{\text{z,j,k}}^m + |U_{\text{z,j,k}}^m|}{2}, \quad \mathcal{U}_{\text{zn}}^m = \frac{U_{\text{z,j,k}}^m - |U_{\text{z,j,k}}^m|}{2};
\]

\[
\mathcal{V}_{\text{zp}}^m = \frac{V_{\text{z,j,k}}^m + |V_{\text{z,j,k}}^m|}{2}, \quad \mathcal{V}_{\text{zn}}^m = \frac{V_{\text{z,j,k}}^m - |V_{\text{z,j,k}}^m|}{2}
\]  
\[\text{and} \quad (3.20)\]

where

\[
\mathcal{U}_{\text{z,j,k}}^m = \frac{U_{\text{z,j,k}}^m + U_{\text{z,j,k+1}}^m}{2}, \quad \mathcal{V}_{\text{z,j,k}}^m = \frac{V_{\text{z,j,k}}^m + V_{\text{z,j,k-1}}^m}{2}
\]  
\[\text{and} \quad (3.21)\]

The horizontal velocity components and the vertical velocity at the free surface are now expressed in terms of the non-hydrostatic pressure.
Analogous to the solution of the three-dimensional governing equations, the non-hydrostatic pressure \( q_{j,k}^{m+1} \) is calculated implicitly using the continuity equation (2.4) discretized in the form

\[
\frac{U_{j+1,k}^{m+1} - U_{j,k}^{m+1}}{R\Delta\lambda \cos \phi_k} + \frac{V_{j,k}^{m+1} \cos(\phi_k + \Delta\phi/2) - V_{j,k-1}^{m+1} \cos(\phi_{k-1} + \Delta\phi/2)}{R\Delta\phi \cos \phi_k} + \frac{w_{r,j,k}^{m+1} - w_{b,j,k}^{m+1}}{D_{j,k}^m} = 0
\]  

(3.22)

Substitution of (3.15), (3.16) and (3.18) into the continuity equation (3.22) gives a linear system of Poisson-type equations at each cell

\[
PL_{j,k} q_{j-1,k}^{m+1} + PR_{j,k} q_{j+1,k}^{m+1} + PB_{j,k} q_{j,k-1}^{m+1} + PT_{j,k} q_{j,k+1}^{m+1} + PC_{j,k} q_{j,k}^{m+1} = QC_{j,k}
\]  

(3.23)

where the coefficients are

\[
PL_{j,k} = \frac{1}{\cos \phi_k} \frac{\Delta t}{2(R\Delta\lambda)^2} \left(1 + A_{j,k}\right)
\]

\[
PR_{j,k} = \frac{1}{\cos \phi_k} \frac{\Delta t}{2(R\Delta\lambda)^2} \left(1 - A_{j+1,k}\right)
\]

\[
PB_{j,k} = \frac{\cos(\phi_{k-1} + \Delta\phi/2)}{\cos \phi_k} \frac{\Delta t}{2(R\Delta\phi)^2} \left(1 + B_{j,k-1}\right)
\]

\[
PT_{j,k} = \frac{\cos(\phi_k + \Delta\phi/2)}{\cos \phi_k} \frac{\Delta t}{2(R\Delta\phi)^2} \left(1 - B_{j,k}\right)
\]

\[
PC_{j,k} = \frac{1}{\cos \phi_k} \frac{\Delta t}{2(R\Delta\lambda)^2} \left[\left(1 + A_{j,k}\right) + \left(1 - A_{j+1,k}\right)\right]
\]

\[
+ \frac{\Delta t}{\cos \phi_k} \frac{\Delta t}{2(R\Delta\phi)^2} \left[\left(1 + B_{j,k-1}\right)\cos(\phi_k + \Delta\phi/2) + \left(1 - B_{j,k}\right)\cos(\phi_{k-1} + \Delta\phi/2)\right]
\]

\[
+ \frac{2\Delta t}{(D_{j,k}^m)^2}
\]  

(3.24)

and the forcing term is
\[ Q_{j,k} = - \frac{\bar{U}_{j+1,k}^{m+1} - \bar{U}_{j,k}^{m+1}}{R\Delta \lambda \cos \phi_k} - \frac{\bar{V}_{j,k}^{m+1} \cos(\phi_k + \Delta \phi/2) - \bar{V}_{j,k-1}^{m+1} \cos(\phi_{k-1} + \Delta \phi/2)}{R\Delta \phi \cos \phi_k} \]

\[ - \frac{w_{s,j,k}^m + w_{h,j,k}^m - 2w_{h,j,k}^{m+1}}{D_{j,k}^m} \]

(3.25)

Assembly of (3.23) at all grid cells gives rise to a matrix equation in the form of

\[ [P] \{ q \} = \{ Q \} \]

(3.26)

which provides the non-hydrostatic pressure at each time step. The Poisson-type equation (3.23) defines the physics of the non-hydrostatic processes through the non-dimensional parameters, \( A_{jk} \) and \( B_{jk} \), and the forcing term \( Q_{j,k} \), which describe the seafloor, free-surface, and velocity gradients in the generation and modification of dispersive waves. The vertical component of the flow, important for sustaining dispersive processes, is imparted through the bottom and surface slopes in terms of the horizontal flow components from the boundary conditions (2.13) and (2.14). As the ocean bottom and surface gradients start to change abruptly at shelf breaks and around steep seamounts and canyons, the coefficients \( A_{jk} \) and \( B_{jk} \) may strongly influence the solution process. The equation type may even change in the runup calculation as the water depth \( h_{j,k} \) varies from positive to negative across the waterline and the values of \( A_{jk} \) and \( B_{jk} \) can be greater than unity. The matrix equation (3.26), in which the matrix \([P]\) is non-symmetric, can be solved by the strongly implicit procedure (SIP) of Stone (1968).

At each time step, the computation starts with the calculation of the hydrostatic solution of the horizontal velocities using (3.7) and (3.8). The non-hydrostatic pressure is then calculated using (3.26) and the horizontal velocities are updated with (3.15) and (3.16) to account for the non-hydrostatic effects. The computation for the non-hydrostatic solution is complete with the calculation of the surface elevation as well as the free surface and the bottom vertical velocities from (3.1), (3.18) and (3.19), respectively.
3.3. Solutions in Cartesian Grids

The numerical formulation in the Cartesian coordinate system can be obtained from the discretized form of the coordinate transformation in (2.31) and (2.32). The grid spacing in the \( x \) and \( y \) directions becomes

\[
\Delta x = R\Delta \lambda \cos \phi_k, \quad \Delta y = R\Delta \phi
\]  (3.27)

The Coriolis terms in horizontal momentum equations (3.7) and (3.8) vanish for modeling of a small coastal region or a laboratory experiment

\[
\begin{align*}
2\Omega + \frac{U^m_{j,k}}{R \cos \phi_k} \left( \frac{\Delta}{\Delta x} \right) = 0 \\
2\Omega + \frac{\bar{U}^m_{y,j,k}}{R \cos(\phi_k + \Delta \phi/2)} \left( \frac{\Delta}{\Delta y} \right) = 0
\end{align*}
\]  (3.28a, 3.28b)

The explicit time integration of the depth-integrated continuity equation becomes

\[
\xi^{m+1}_{j,k} = \xi^{m}_{j,k} - \Delta t \left( \frac{FLX_{j+1,k} - FLX_{j,k}}{\Delta x} \right) - \Delta t \left( \frac{FLY_{j,k} - FLY_{j,k-1}}{\Delta y} \right)
\]  (3.29)

where the flux terms are estimated from (3.2). The integration of the hydrostatic terms in the momentum equations gives

\[
\begin{align*}
\tilde{U}^{m+1}_{j,k} &= U^m_{j,k} - \frac{g\Delta t}{\Delta x} \left( \xi^{m}_{j,k} - \xi^{m}_{j-1,k} \right) - \frac{\Delta t}{\Delta x} U^m_{p} \left( U^{m}_{j+1,k} - U^{m}_{j,k} \right) - \frac{\Delta t}{\Delta x} U^m_{n} \left( U^{m}_{j,k+1} - U^{m}_{j,k} \right) \\
&- \frac{\Delta t}{\Delta y} \tilde{V}^m_{x,p} \left( U^{m}_{j,k} - U^{m}_{j,k-1} \right) - \frac{\Delta t}{\Delta y} \tilde{V}^m_{x,n} \left( U^{m}_{j,k+1} - U^{m}_{j,k} \right) \\
&- n^2 g \frac{\Delta t U^m_{j,k} \sqrt{\left( U^m_{j,k} \right)^2 + \left( \tilde{V}^m_{x,j,k} \right)^2}}{\left( D^{m}_{j-1,k} + D^{m}_{j,k} \right)^{3/2}}
\end{align*}
\]  (3.30)
The momentum-conserved advection scheme (3.9)-(3.11) provides the advective speeds to handle flow discontinuities.

The time integration of the non-hydrostatic terms in the horizontal momentum equations becomes

\[
\begin{align*}
U_{j,k}^{m+1} &= \tilde{U}_{j,k}^{m+1} - \frac{\Delta t}{\Delta x} A_{j,k} \left( \frac{q_{j,k}^{m+1} + q_{j-1,k}^{m+1}}{2} \right) - \frac{\Delta t}{\Delta y} \frac{\Delta t}{\Delta x} \left( \frac{q_{j,k}^{m+1} - q_{j-1,k}^{m+1}}{2} \right) \\
\nu_{j,k}^{m+1} &= \tilde{\nu}_{j,k}^{m+1} - \frac{\Delta t}{\Delta y} B_{j,k} \left( \frac{q_{j,k}^{m+1} + q_{j,k+1}^{m+1}}{2} \right) - \frac{\Delta t}{\Delta y} \frac{\Delta t}{\Delta y} \left( \frac{q_{j,k}^{m+1} - q_{j,k+1}^{m+1}}{2} \right)
\end{align*}
\]

where

\[
A_{j,k} = \frac{(\rho_{j,k} - h_{j,k}) - (\rho_{j-1,k} - h_{j-1,k})}{(D_{j,k}^{m} + D_{j,k+1}^{m})} \quad B_{j,k} = \frac{(\rho_{j,k+1} - h_{j,k+1}) - (\rho_{j,k} - h_{j,k})}{(D_{j,k}^{m} + D_{j,k+1}^{m})}
\]

The free surface and seafloor vertical velocities are given by

\[
w_{s,j,k}^{m+1} = w_{s,j,k}^{m} + \left( w_{b,j,k}^{m+1} - w_{b,j,k}^{m} \right) + \frac{2\Delta t}{D_{j,k}^{m}} q_{j,k}^{m+1}
\]

\[
w_{h,j,k}^{m+1} = w_{h,j,k}^{m} + \left( h_{j,k}^{m} - h_{j-1,k}^{m} \right) - \frac{\Delta t}{\Delta x} \frac{\Delta t}{\Delta y} \left[ \frac{h_{j,k}^{m} - h_{j,k+1}^{m}}{\Delta y} - \frac{h_{j,k}^{m} - h_{j,k-1}^{m}}{\Delta y} - \frac{h_{j,k}^{m} - h_{j,k+1}^{m}}{\Delta y} \right]
\]

where the advective speeds are given by (3.20) and (3.21). The horizontal velocity and the free surface vertical velocity, which are expressed in terms of the non-hydrostatic
pressure in (3.32) (3.33) and (3.35), update the hydrostatic solution to account for wave dispersion effects.

The non-hydrostatic pressure $q_{j,k}^{m+1}$ is calculated implicitly using the three-dimensional continuity equation (2.4) along with the coordinate transformation (2.31) and discretized in the form:

$$
\frac{U_{j+1,k}^{m+1} - U_{j,k}^{m+1}}{\Delta x} + \frac{V_{j,k}^{m+1} - V_{j,k-1}^{m+1}}{\Delta y} + \frac{W_{j,k}^{m+1} - W_{b,j,k}^{m+1}}{D_{j,k}^m} = 0
$$

(3.37)

The linear system of Poisson-type equations (3.23) is obtained by substitution of (3.32), (3.33) and (3.35) into the continuity equation (3.37). The resulting coefficients and the forcing term in the Cartesian grid are

$$
PL_{j,k} = \frac{\Delta t}{2\Delta x^2} \left( -1 + A_{j,k} \right)
$$

$$
PR_{j,k} = \frac{\Delta t}{2\Delta x^2} \left( -1 - A_{j+1,k} \right)
$$

$$
PB_{j,k} = \frac{\Delta t}{2\Delta y^2} \left( -1 + B_{j,k-1} \right)
$$

$$
PT_{j,k} = \frac{\Delta t}{2\Delta y^2} \left( -1 - B_{j,k} \right)
$$

$$
PC_{j,k} = \frac{\Delta t}{2\Delta x^2} \left[ (1 + A_{j,k}) + (1 - A_{j+1,k}) \right] + \frac{\Delta t}{2\Delta y^2} \left[ (1 + B_{j,k-1}) + (1 - B_{j,k}) \right] + \frac{2\Delta t}{(D_{j,k}^m)^2}
$$

(3.38)

$$
Q_{j,k} = \frac{\bar{U}_{j+1,k}^{m+1} - \bar{U}_{j,k}^{m+1}}{\Delta x} - \frac{\bar{V}_{j,k}^{m+1} - \bar{V}_{j,k-1}^{m+1}}{\Delta y} - \frac{w_{j,k}^m + w_{b,j,k}^m - 2w_{b,j,k}^{m+1}}{D_{j,k}^m}
$$

(3.39)

The solution procedure is identical to that followed for the spherical coordinate system. The computation starts with the hydrostatic solution of the horizontal velocities using (3.30) and (3.31). The matrix equation (3.26) with the coefficients and forcing term from (3.38) and (3.39) provides non-hydrostatic pressure and (3.32) and (3.33) updates the
horizontal velocity to account for the non-hydrostatic effects. The non-hydrostatic solution is complete with the calculation of the surface elevation and the vertical velocities from (3.29), (3.35), and (3.36).
CHAPTER 4
IMPLEMENTATION FOR TSUNAMI MODELING

The depth-integrated, non-hydrostatic model provides a general framework to describe dispersion and breaking of ocean waves over variable bottom. This section discusses its implementation with a wet-dry moving boundary condition, a two-way grid-nesting scheme, and a bathymetry-smoothing strategy for modeling of the tsunami evolution processes from generation to runup. The wet-dry moving boundary condition tracks the interface between water and dry land to model the runup and drawdown processes at the coast. The proposed two-way, grid-nesting scheme utilizes a flexible indexing system that enables adaptation of inter-grid boundaries to topographic features for optimal resolution and computational efficiency. The smoothing scheme generalizes the stability requirements of Horrillo et al. (2006) to remove small-scale bathymetric features in relation to the water depth that cannot be resolved by non-hydrostatic or Boussinesq-type models.

4.1 Wet-Dry Moving Boundary Condition

For inundation or runup calculations, special numerical treatments are necessary to describe the moving waterline in the swash zone. The present non-hydrostatic model tracks the interface between wet and dry cells using the approach of Kowalik and Murty (1993), which was originally developed for hydrostatic flow. The basic idea is to extrapolate the numerical solution from the wet region onto the beach. The non-hydrostatic pressure is set to be zero at the wet cells along the wet-dry interface to conform to the physical problem and to improve stability of the scheme.
The moving waterline scheme provides an update of the wet-dry interface as well as the associated flow depth and velocity at the beginning of every time step. A marker $CELL_{j,k}^m$ first updates the wet-dry status of each cell based on the flow depth and surface elevation. If the flow depth $D_{j,k}^m$ is positive, the cell is under water and $CELL_{j,k}^m = 1$, and if $D_{j,k}^m$ is zero or negative, the cell is dry and $CELL_{j,k}^m = 0$. This captures the retreat of the waterline in an ebb flow. The surface elevation along the interface then determines any advancement of the waterline. For flows in the positive $\lambda$ or $x$ direction, if $CELL_{j,k}^m$ is dry and $CELL_{j-1,k}^m$ is wet, $CELL_{j,k}^m$ is reevaluated as

$$CELL_{j,k}^m = 1 \text{ (wet) if } \zeta_{j-1,k}^m > -h_{j,k}$$

$$CELL_{j,k}^m = 0 \text{ (dry) if } \zeta_{j-1,k}^m \leq -h_{j,k}$$

If $CELL_{j,k}^m$ becomes wet, the scheme assigns the flow depth and velocity at the cell as

$$D_{j,k}^m = \zeta_{j-1,k}^m + h_{j,k}, \ U_{j,k}^m = U_{j-1,k}^m$$

The marker $CELL_{j,k}^m$ is then updated for flows in the negative direction. The same procedures are implemented in the $\phi$ or $y$ direction to complete the wet-dry status of the cell. If water flows into a new wet cell from multiple directions, the flow depth is averaged.

Once the wet-dry cell interface is open by setting $CELL_{j,k}^m = 1$, the flow depth $D_{j,k}^m$ and velocity $(U_{j,k}^m, V_{j,k}^m)$ are assigned to the new wet cell to complete the update of the wet-dry interface at time step $m$. The surface elevation $\zeta_{j,k}^{m+1}$ and the flow velocity $(U_{j,k}^{m+1}, V_{j,k}^{m+1})$ over the computational domain are obtained from integration of the momentum and continuity equations along with the implicit solution of the non-hydrostatic pressure as outlined in Sections 3.1 and 3.2. The moving waterline scheme is then repeated to update the wet-dry interface at the beginning of the $(m+1)$ time step. This approach
remains stable and robust for the non-hydrostatic flows without artificial dissipation mechanisms.

4.2 Grid-nesting Scheme

Most grid-nesting schemes for hydrostatic models use a system of rectangular grids with the fluxes as input variables to a fine inner grid and the surface elevation as output to the outer grid (e.g., Liu et al., 1995b; Goto et al., 1997; Wei et al., 2003; Yamazaki et al., 2006; and Sánchez and Cheung, 2007). A non-hydrostatic model needs to input the velocity, surface elevation as well as the non-hydrostatic pressure to ensure propagation of breaking and dispersion waves across inter-grid boundaries. Linear interpolation of these quantities from the outer grid along the inter-grid boundary provides the input to the inner grid. Figure 4.1 illustrates the grid setup and data transfer in the two-way nesting scheme. The outer and inner grids align at the cell centers, where they share the same water depth. The scheme allows the flexibility to define the inter-grid boundaries at any location within the inner grid network analogous to the quadtree and adaptive grids (e.g., Park and Borthwick, 2001; Liang et al., 2008; and George, 2010). The surface elevation and non-hydrostatic pressure are input at the cell centers as indicated by the red dots. While the tangential component of the velocity is applied along the inter-grid boundary, the normal component is applied on the inboard side of the boundary cells as indicated by the red dashes. Existing grid-nesting schemes, which only input the normal velocity, cannot handle propagation of discontinuities and Coriolis effects across inter-grid boundaries.

Figure 4.2 shows a schematic of the solution procedure with the two-way grid-nesting scheme. The time step \( \Delta t_1 \) at the outer grid must be divisible by the inner-grid time step \( \Delta t_2 \). The calculation begins at time \( t \) with the complete solutions at both the outer and
Figure 4.1 Schematic of a two-level nested grid system. (a) Nested-grid configuration. (b) Close-up view of the inter-grid boundary and data transfer protocol.
inner grids. The time-integration procedure provides the solution over the outer grid at 
(t + Δt1). Linear interpolation of the horizontal velocity, surface elevation, and non-
hydrostatic pressure in time and space provides the input boundary conditions to the inner 
grid. The time-integration procedure then computes the inner grid solution at Δt2 
increments until (t + Δt1). The hydrostatic solution is computed explicitly from the input 
surface elevation and horizontal velocity components. The non-hydrostatic solution is 
implicit and requires reorganization of the matrix equation (3.26) for the input boundary 
conditions. The simple dispersive terms with first-order derivatives allows 
implementation of the Dirichlet condition to enable wave dispersion across inter-grid 
boundaries. After the computation in the inner grid reaches (t + Δt1), the surface elevation

![Figure 4.2 Schematic of two-way grid-nesting and time-integration scheme.](image)

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at the outer grid is then updated with the average value from the overlapping inner grid cells to complete the procedure. This feedback mechanism is similar to the grid-nesting scheme of Goto et al. (1997) for hydrostatic models.

The implementation of the Dirichlet boundary condition requires modification of the Poisson-type equation (3.23) at the cells along the inter-grid boundary. For example, if the non-hydrostatic pressure is input from the left (west), the corresponding pressure term becomes part of the forcing on the right hand side as

$$\begin{align*}
PR_{j,k} q_{j+1,k}^{m+1} + PB_{j,k} q_{j,k-1}^{m+1} + PT_{j,k} q_{j,k+1}^{m+1} + PC_{j,k} q_{j,k}^{m+1} &= Q_{j,k} - PL_{j,k} q_{j-1,k}^{m+1} \\
(4.1)
\end{align*}$$

Similarly, if the non-hydrostatic pressure is defined at the bottom (south), (3.23) becomes

$$\begin{align*}
PL_{j,k} q_{j-1,k}^{m+1} + PR_{j,k} q_{j+1,k}^{m+1} + PT_{j,k} q_{j,k+1}^{m+1} + PC_{j,k} q_{j,k}^{m+1} &= Q_{j,k} - PB_{j,k} q_{j,k-1}^{m+1} \\
(4.2)
\end{align*}$$

When the non-hydrostatic pressure is input from the left (west) and bottom (south), the Dirichlet boundary condition gives rise to

$$\begin{align*}
PR_{j,k} q_{j+1,k}^{m+1} + PT_{j,k} q_{j,k+1}^{m+1} + PC_{j,k} q_{j,k}^{m+1} &= Q_{j,k} - PL_{j,k} q_{j-1,k}^{m+1} - PB_{j,k} q_{j,k-1}^{m+1} \\
(4.3)
\end{align*}$$

Similar equations can be defined for cells with input from the right (east) and top (north). This approach allows implementation of the Dirichlet boundary condition along irregular inter-grid boundaries to construct the matrix equation (3.26), from which the non-hydrostatic pressure in the inner grid can be determined.

### 4.3 Depth-dependent Smoothing Scheme

The present depth-integrated, non-hydrostatic model is generally stable in practical applications. The implicit solver for the non-hydrostatic pressure, however, does not always converge when applied to small bathymetric features in relation to the water depth. Horrillo et al. (2006) discussed similar convergence issues with the dispersive term in the classical Boussinesq model and derived the stability condition $\Delta x > 1.5h$ from their
numerical formulation. Løvholt and Pederson (2008) also showed Boussinesq-type models are prone to instability over localized, steep bottom gradients at high-resolution computations. They pointed out that the instability is probably due to the dispersive term. If the instability is inherent in the theoretical formulation, it is difficult to resolve the issue through refinements of numerical schemes. Smoothing of the bathymetry appears to be a typical solution to stability problems associated with Boussinesq-type models, even though Plant et al. (2009) pointed out such numerical treatment might alter wave fields in the near-shore region.

Preliminary numerical tests have shown the present model could maintain stability with properly selected grid sizes in relation to the water depth. Such an approach, however, is not feasible for abrupt bathymetric changes over deep trenches and volcanic seamounts in the open ocean. A depth-dependent Gaussian function is considered here to resolve stability issues arising from small bathymetric features in deep water and to minimize unnecessary alternations of the bathymetry in near-shore waters. The smoothing scheme has a variable search diameter in terms of the water depth $h_{j,k}$ as

$$\Delta D_{j,k} = \alpha h_{j,k} \tag{4.4}$$

where $\alpha > 1.5$ in accordance to the stability criterion of Horrillo et al. (2006). In the implementation, the smoothed water depth is given by

$$h_{j,k} = \sum_{j=j-n/2}^{j+n/2} \sum_{k=k-n/2}^{k+n/2} w_{j,k} h_{o_{j,k}} \tag{4.5}$$

where $w_{j,k}$ is a weight function, $h_{o_{j,k}}$ is the original bathymetry at $(\tilde{j},\tilde{k})$, and $n$ is the number of grid cells within the search diameter $\Delta D_{j,k}$. The weight function is expressed as

$$w_{j,k} = G_{j,k} \left( \sum_{j=j-n/2}^{j+n/2} \sum_{k=k-n/2}^{k+n/2} G_{j,k} \right)^{-1} \tag{4.6}$$
in which the Gaussian function is defined as

\[ G_{j,k} = \frac{2}{\pi \Delta D_{j,k}^2} \exp\left(-\frac{2r_{j,k}^2}{\Delta D_{j,k}^2}\right) \]  \hspace{1cm} (4.7)

where \( r_{j,k} \) is the distance between the points \((\tilde{j}, \tilde{k})\) and \((j, k)\). This scheme smooths features with characteristic dimensions less than \( \alpha h \) to accommodate smaller computational grid cells.

In addition, the present model allows the use of a series of nested grids over complex bathymetry and topography. The grid resolution changes abruptly from one level to the next. The resolution of the relief data, however, needs to transition gradually across inter-grid boundaries since interpolated data at different grid resolution could have noticeable discrepancies leading to instabilities in the nesting scheme. The time-stepping scheme integrates the solution at the outer and inner grids separately and passes information at the inter-grid boundary through interpolation. We interpolate the outer grid bathymetry at the inner grid cells adjacent to the inter-grid boundary and gradually transition the resolution of the bathymetry to that of the inner grid over a distance equal to two outer grid cells. This procedure becomes imperative in coastal water when the inter-grid boundary intersects the coastlines and interpolation of flow parameters occurs across the wet-dry boundary.

4.4 NEOWAVE

The non-hydrostatic formulation with dynamic seafloor deformation, the momentum-conserved advection scheme improved by the upwind flux approximation, as well as the grid-nesting scheme are implemented in the finite difference, nonlinear shallow-water model of Kowalik et al. (2005) for tsunami modeling. The resulting program, NEOWAVE (Non-hydrostatic Evolution of Ocean WAVE), is written with a modular
structure in FORTRAN 90 for serial computation. The code is based on the spherical coordinate system for basin–wide applications. The coordinate transformation with (3.27) and (3.28) allows applications with regional coastal problems as well as laboratory experiments in Cartesian grids. The model currently can accommodate up to five levels of nested grids in either coordinate system. This provides a systematic application environment for a range of problems using one single model package.

The modular structure of NEOWAVE allows selection from a variety of numerical schemes in building a hydrostatic or non-hydrostatic model for a specific application. Figure 4.3 shows the options in NEOWAVE. All the options are available in both the non-hydrostatic and hydrostatic modules. The fluxes in the continuity equation can be computed using the upwind scheme (3.2) or the second-order scheme (3.4). The

Figure 4.3 Schematic of module structure of NEOWAVE.
advective speed in the horizontal momentum equations can be solved with the momentum-conserved advection scheme with the upwind flux approximation (3.9)-(3.11), the momentum-conserved advection scheme (3.12)-(3.14) of Stelling and Duinmeijer (2003), or the standard non-conservative scheme (3.3). The effect of wave dispersion can be investigated through comparison of non-hydrostatic and hydrostatic solutions. The effect of wave breaking and bore development can be examined by comparing the solutions with and without the momentum-conserved advection scheme. These functionalities in the model package allow investigation of various numerical schemes in describing specific flow physics.

NEOWAVE is becoming a community model with multiple developers and users around the world. The development work and model update are coordinated at the Department of Ocean and Resources Engineering, University of Hawaii. The source code and manual are available to the public for non-commercial use.
CHAPTER 5
DISPERSION AND NESTED GRIDS

This section verifies the capability of NEOWAVE in describing dispersive waves and their propagation through a nested grid system. A series of numerical experiments involving solitary wave propagation in a channel, sinusoidal wave transformation over a submerged bar, and \( N \)-wave transformation and runup over realistic bathymetry provide an assessment of the wave dispersion characteristics and verification of the grid–nesting scheme. In addition, the effectiveness of the upwind flux scheme is examined in the sinusoidal wave transformation experiment.

5.1 Solitary Wave Propagation in a Channel

The propagation of a solitary wave in uniform water depth represents a delicate balance between nonlinear steepening and dispersive spreading. This is also an analytical solution of the classical Boussinesq equations that all advanced dispersive wave models must be able to describe. As a result, the numerical experiment of solitary wave propagation in a channel of constant depth has been a standard test for dispersive wave models (e.g., Stelling and Zijlema, 2003; Walters, 2005; and Roeber et al., 2010). The numerical solution should maintain the solitary waveform and celerity through propagation in an inviscid fluid.

The first numerical experiment involves a 2500-m long and 10-m deep channel with radiation conditions at both ends. The initial condition corresponds to a 2-m high solitary wave at \( x = 100 \) m. The computation uses \( \Delta x = \Delta y = 1.0 \) m, \( \Delta t = 0.05 \) sec, and a Manning’s roughness coefficient \( n = 0.0 \) for the inviscid flow. Figure 5.1 shows the initial solitary wave and the computed waveforms along the channel at 5, 60, 120, and
180 sec. The wave height decreases slightly at the very beginning due to the use of an analytical solution as the initial condition. The computed waveform stabilizes with a maximum surface elevation of 1.92 m after $t \approx 5$ sec. The horizontal dotted line at $\zeta = 1.92$ m indicates that the wave height remains steady for the reminder of the simulation. The computed waveform maintains its symmetry without noticeable trailing waves after propagating for 180 sec in the channel. The ability to maintain the solitary waveform derives from the non-hydrostatic terms in the formulation. Numerical experiments with the upwind flux approximation (3.2) replaced by the second-order scheme (3.4) or without the momentum-conserved advection scheme (3.9)-(3.11) yielded very similar

![Figure 5.1 Solitary wave profiles along a channel with constant water depth.](image)
results, which are not presented here for brevity. Refinement of the computational grid, however, diminished the initial reduction of the wave height, confirming that as a numerical artifact.

The numerical experiment of solitary wave propagation also provides a critical test for the grid-nesting scheme. The model must accurately transfer the flow kinematics and non-hydrostatic pressure and balance the nonlinear and dispersive effects across nested grids of different resolution to maintain the solitary waveform. In the second numerical experiment, a solitary wave propagates in a 45 m long, 25 m wide, and 0.5 m deep channel with two levels of nested grids. The initial conditions correspond to a 0.05-m high solitary wave at $x = 7.5$ m. Figure 5.2 shows a sequence of the computed waveforms in their original resolution as the solitary wave propagates through the inner grid. The numerical experiment uses 20-cm resolution for the level-1 outer grid and 4-cm for the level-2 inner grid. The inner grid configuration allows testing of solitary wave propagation at different directions with respect to the inter-grid boundary. The solitary wave enters the level-2 grid around $t = 2.5$ sec and leaves at 15.0 sec. Even though the inter-grid boundary is set oblique to the wave direction, the solitary wave passes through the nested-grid smoothly with invisible surface disturbance and leaves no residual oscillation near the inter-grid boundary, thereby verifying the capability of the grid-nesting scheme for dispersive wave propagation.
Figure 5.2 Propagation of solitary wave across two levels of nested grids in a channel of constant depth. Dark blue indicates the level-1 grid and light blue denotes the level-2 grid results.
5.2 Sinusoidal Wave Propagation over a Bar

Beji and Battjes (1993) and Luth et al. (1994) conducted a laboratory experiment to examine sinusoidal wave propagation over a submerged bar. Figure 5.3(a) shows the experiment setup in a 37.7-m long, 0.8-m wide, and 0.75-m high wave flume. A hydraulically driven, piston-type wave generator is located at the left side of the flume and a 1:25 plane beach with coarse material is placed at the right side to serve as a wave absorber. The submerged trapezoidal bar is 0.3 m high with front slope of 1:20 and lee slope of 1:10. The computational domain in Figure 5.3(b) is 35 m long and 0.4 m deep and is discretized with $\Delta x = \Delta y = 1.25$ cm and $\Delta t = 0.01$ sec. Surface roughness is unimportant in this experiment and a Manning’s coefficient $n = 0.0$ is used. We consider the test case with 1-cm incident wave amplitude and 2.02-sec wave period that

![Figure 5.3 Definition sketch of wave transformation over a submerged bar. (a) Laboratory setup. (b) Numerical setup. ○, gauge locations.](image)
corresponds to the water depth parameter $kh = 0.67$. The incident sinusoidal waves are generated at the left boundary and the radiation boundary condition is imposed on the right. The free surface elevations are output at eight gauge locations over and behind in accordance to the laboratory experiment.

Figure 5.4 shows the computed and recorded waveforms at the eight gauges. The measured data at gauge 4 provides a reference for adjustment of the timing of computed waveforms. The present model with either the upwind flux approximation (3.2) or the

![Figure 5.4 Comparison of computed and measured free surface elevations over and behind a submerged bar. ○, laboratory data of Beji and Battjes (1993); - - - - (red), non-hydrostatic model with upwind flux scheme; - - - - (blue), second order scheme.](image)
second-order scheme (3.4) in the continuity equation reproduces the wave transformation at gauge 4 over the front slope and at gauge 5 immediately behind the front slope. The computed results maintain good agreement with the laboratory data at gauges 6 to 8 over the crest and the lee slope, where the waveform undergoes significant transformation with high frequency dispersion. Noticeable discrepancies arise between the computed and recorded waveforms over the flat bottom behind the bar, where the laboratory data from gauges 9 to 11 shows evidence of super-harmonics around 1 sec and 0.67 sec periods, which correspond to $kh = 1.7$ and $3.6$, respectively. The third-order waves with $kh = 3.6$ behind the bar are the source of dispersion errors as inferred from Figure 2.2.

Overall, the upwind flux approximation (3.2) and the second-order scheme (3.4) in the present model provide the same or slightly better results compared to Stelling and Zijlema (2003) and Walters (2005). The present and previously computed waveforms are almost identical, when the depth-integrated models are within the applicable range of the non-hydrostatic approximation. The high-frequency waves behind the bar exceed the applicable range of a weakly dispersive model. Under these critical conditions, the second-order scheme provides larger wave amplitudes, which can be seen at gauges 7 to 10, especially at gauge 9. This alludes to the importance of the upwind flux approximation in maintaining numerical stability of the model especially when implemented outside its applicable range.

5.3 N-wave Transformation and Runup

Matsuyama and Tanaka (2001) conducted a laboratory study at the Central Research Institute for Electric Power Industry (CRIEPI) to investigate the nearshore wave dynamics of the 1993 Hokkaido Nansei-Oki Tsunami. Eyewitness reported initial withdraw of the water followed by a large tsunami wave at the coast causing 31.7 m of
runup near Monai Valley, Japan. Such observations fit the description of the leading-depression $N$-wave, which is often used in laboratory studies of tsunami impact close to the source. The CRIEPI wave flume is 205 m long, 3.4 m wide, and 6 m high with reflective sidewalls and a hydraulic, piston-type wave maker capable of generating $N$-waves. The 1:400-scale coastal relief model around Monai Valley was constructed of painted plywood and installed approximately 140 m from the wave generator. The initial $N$-wave was relatively long with a very gentle profile and dispersed into a series of short-period waves over the coastal relief model in the experiment. The measured water level data and runup from the experiment allow validation of the present grid-nesting scheme in describing generation of dispersive waves across inter-grid boundaries.

The grid-nesting scheme describes wave dynamics at resolution compatible with the physical process and spatial scale for optimization of computational efficiency. Figure 5.5(a) shows the numerical experiment setup over the 5.475-m long and 3.4-m wide relief model at 1.4 cm resolution (5.6 m full scale). The level-1 grid covers the entire panel at 2.5 cm resolution (10 m full scale) and the level-2 grid at 1.25-cm resolution (5 m full scale) provides a more detailed description of the nearshore wave processes in the outlined area from Monai Valley to the 0.035-m (14-m full scale) depth contour. A Manning’s coefficient $n = 0.012$ describes the surface roughness of the painted plywood model (Chaudhry, 1993). Figure 5.5(b) shows the input $N$-wave profile at the left boundary with 0.135 m (54 m full scale) water depth. The incident waves shoal over a plane slope before refracting and diffracting around a small island on a shallow bank. Figure 5.6(a) shows very good agreement of the computed surface elevations with the laboratory measurements at three gauges behind the island. The model reproduces the small-amplitude dispersive waves generated by reflection from the coast with a minor
phase lag, but the results are better than those from the extended Boussinesq model of Nwogu (2008) in dispersive wave estimation. We recomputed the results with a uniform grid at the same 1.25-cm resolution (5 m full scale) as the level-2 nested grid over the entire domain and obtained almost identical results as shown in Figure 5.6(b).
Figure 5.6 Time series of surface elevation at gauges in Monai Valley experiment. (a) Comparison of measurements with nested-grid solution. (b) Comparison of nested and uniform grid solutions. —— (black), laboratory data of Matsuyama and Tanaka (2001); —— (red), grid-nesting solution; —— (blue), uniform grid solution.

Table 5.1. Recorded runup for the six trials from Matsuyama and Tanaka (2001).

<table>
<thead>
<tr>
<th>Trial No.</th>
<th>Maximum $R_{\text{max}}$ (cm) (Full scale in m)</th>
<th>$y = 2.2062$ m</th>
<th>$y = 2.32$ m</th>
</tr>
</thead>
<tbody>
<tr>
<td>209_105</td>
<td>8.75 (35.0)</td>
<td>5.25 (21.0)</td>
<td>5.25 (21.0)</td>
</tr>
<tr>
<td>209_106</td>
<td>9.00 (36.0)</td>
<td>5.75 (23.0)</td>
<td>5.50 (22.0)</td>
</tr>
<tr>
<td>209_107</td>
<td>8.00 (32.0)</td>
<td>5.50 (22.0)</td>
<td>5.50 (22.0)</td>
</tr>
<tr>
<td>210_101</td>
<td>9.00 (36.0)</td>
<td>6.50 (26.0)</td>
<td>5.75 (23.0)</td>
</tr>
<tr>
<td>210_102</td>
<td>10.00 (40.0)</td>
<td>6.75 (27.0)</td>
<td>5.75 (23.0)</td>
</tr>
<tr>
<td>210_103</td>
<td>9.00 (36.0)</td>
<td>6.50 (26.0)</td>
<td>5.75 (23.0)</td>
</tr>
<tr>
<td>mean</td>
<td>8.97 (35.8)</td>
<td>6.04 (24.2)</td>
<td>5.58 (22.3)</td>
</tr>
</tbody>
</table>
Table 5.1 lists the runup measurements along $y = 2.2062$ and 2.32 m as well as the maximum value inside Monai Valley from a series of tests in the laboratory experiment of Matsuyama and Tanaka (2001). Figure 5.7 shows the computed runup and inundation from the nested and uniform grids along with the range and mean value of the recorded runup. The computed results show good agreement with the measured data despite its uncertainty. Both the nested and the uniform grid solutions are almost identical indicating

Figure 5.7 Runup and Inundation comparisons. (a) Runup, (b) Inundation. ○, laboratory data of Matsuyama and Tanaka (2001); (red), grid-nesting solution of Level-2 grid at 1.25-cm resolution; —- - - (blue), single grid solution at 1.25-cm resolutions; —— (black), topography contours at 0.0125-m intervals; —- - - (grey), bathymetry contours at 0.0125-m intervals.
the proposed grid-nesting scheme appropriately transfers information across the inter-grid boundary and provide results of the same accuracy as the uniform grid solution. The minor discrepancies near the inter-grid boundaries are due to transition of the bathymetry from the outer to the inner grid. Comparisons of the surface elevation, runup, and inundation from the Moani Valley experiment verify the proposed grid-nesting scheme in handling dispersive wave and runup processes over complex nearshore bathymetry and topography.
NEOWAVE uses a momentum conservation scheme to model wave breaking as bores or hydraulic jumps without the use of artificial dissipation models. The scheme, which does not require calibration, accounts for energy dissipation across flow discontinuities. The benchmark laboratory experiments of solitary wave transformation over a plane beach and a conical island involve energetic breaking waves in the runup and drawdown processes. Numerical experiments of these tests provide a systematic examination of the model performance with and without the momentum conservation scheme.

### 6.1 Solitary Wave Runup on a Plane Beach

Hall and Watts (1953), Li and Raichlen (2002), and Synolakis (1987) conducted series of laboratory experiments for solitary wave transformation and runup on a plane beach. These experiments, which cover a wide range of non-breaking and breaking waves, have become an accepted test case for validation of runup models (Titov and Synolakis, 1995; Lynett et al., 2002; Li and Raichlen, 2002; Wei et al., 2006; and Roeber et al., 2010). Figure 6.1 provides a schematic of the experiments with $A$ indicating the incident solitary wave height.

![Figure 6.1 Definition sketch of solitary wave runup on a plane beach.](image)
wave height, $\beta$ the beach slope, and $R$ the runup. Following Titov and Synolakis (1995), the solitary wave is initially at a half wavelength from the toe of the beach in the numerical experiment. The approximate wavelength of a solitary wave is given by

$$L = \frac{2}{k} \arccosh \left( \frac{1}{\sqrt{0.05}} \right)$$

(6.1)

in which the wave number $k = \sqrt{3A/4h^3}$. In the numerical experiment, we use $\Delta x/h = 0.125$ and a Courant number $Cr = 0.2$. Surface roughness becomes important for runup over gentle slopes and a Manning’s coefficient $n = 0.01$ describes the surface condition of the smooth glass beach in the laboratory experiments (Chaudhry, 1993).

Titov and Synolakis (1995) presented a series of surface profiles with a beach slope of 1:19.85 and solitary wave heights of up to $A/h = 0.3$. Initial testing with this experiment reiterates and furthers the findings from the experiments of sinusoidal wave transformation in Section 5.2. The upwind flux approximation of the surface elevation is essential in maintaining stability of the depth-integrated non-hydrostatic model, especially when flow discontinuities associated with breaking waves and hydraulic jumps develop. The second-order scheme produces spurious waves near the discontinuity that lead to development of instabilities in most of the tests. As a result, this test case uses the upwind flux approximation in the model and examines the treatment of the advective terms in the momentum equations by presenting results with and without the momentum-conserved advection approximation (3.9)-(3.11). When this scheme is off, the advective terms are computed directly from (3.3).

Figure 6.2 shows a comparison of the measured profiles with the two sets of numerical results for the test case with the solitary wave height $A/h = 0.3$. The laboratory data shows wave breaking between $t(g/h)^{1/2} = 20$ and 25 as the solitary wave reaches the beach and development of a hydraulic jump at $t(g/h)^{1/2} = 50$ when the water recedes from the beach.
Figure 6.2 Surface profiles of a solitary wave transformation on a 1:19.85 plane beach with $A/h = 0.3$. ○, laboratory data of Titov and Synolakis (1995); (red), non-hydrostatic model with momentum-conserved advection; (blue), without momentum-conserved advection.

Both numerical solutions show very good agreement with the laboratory data as the solitary wave shoals to its maximum height at $t(g/h)^{1/2} = 20$. The momentum-conserved advection scheme reproduces the subsequent wave breaking without the use of predefined criteria and matches the surface elevation and runup on the beach. Without the momentum-conserved advection, the model cannot reproduce the surface profile at
\((g/h)^{\frac{1}{2}} = 25\) immediately after wave breaking and underestimates the surface elevation on the beach and eventually the runup. Both solutions describe the surface elevation reasonably well during the drawdown process. A minor discrepancy on the location of the hydraulic jump occurs around the peak of the return flow at \(t(g/h)^{\frac{1}{2}} = 55\). The finite volume model of Wei et al. (2006) also produces a similar discrepancy with the laboratory data. This may be attributed to the three-dimensional flow structure that is not amenable to depth-integrated solutions. The agreement resumes as the speed of the return flow decreases demonstrating the resilience of the model.

Figure 6.3 shows the computed and measured runup \(R/h\) as a function of the solitary wave height \(A/h\) for beach slopes of 1:5.67, 1:15, and 1:19.85. The measured data shows a bilinear distribution with the two branches representing the non-breaking and breaking regimes separated by a transition. Figure 6.3(a) shows good agreement of the two solutions with the laboratory data for non-breaking and breaking wave runup on the 1:5.67 slope. This steep slope most likely produces surging wave breakers that are amenable to non-hydrostatic models without special treatments to the momentum equations. Wave breaking becomes more energetic and the resulting surface elevation becomes discontinuous as the beach slope decreases. In Figure 6.3(b) and 6.3(c), the computed runup from the momentum-conserved advection approximation shows excellent agreement with the laboratory data for both non-breaking and breaking waves. Without the momentum-conserved advection, the solution reproduces the runup in the non-breaking and transition regimes, but underestimates the measured runup for breaking waves with \(A/h > 0.1\). This shows that implementation of the momentum-conserved advection scheme in a non-hydrostatic model can capture discontinuous flows associated with energetic wave breaking and describe the subsequent runup on the beach without an empirical dissipation term.
Figure 6.3 Solitary wave runup on a plane beach as a function of incident wave height. (a) 1/5.67 (Hall and Watts, 1953). (b) 1/15 (Li and Raichlen, 2002). (c) 1/19.85 (Synolakis, 1987). ○, laboratory data; (red), non-hydrostatic model with momentum-conserved advection; - - - - (blue), without momentum-conserved advection.
6.2 Solitary Wave Runup on a Conical Island

Briggs et al. (1995) conducted a large-scale laboratory experiment to investigate solitary wave runup on a conical island. The collected data has become a standard for validation of runup models (Liu et al., 1995a; Titov and Synolakis, 1998; Chen et al., 2000; Lynett et al., 2002; and Wei et al., 2006). Figure 6.4 shows a schematic of the experiment. The basin is 25 m by 30 m. The circular island has the shape of a truncated cone with diameters of 7.2 m at the base and 2.2 m at the crest. The island is 0.625 m high and has a side slope of 1:4. The surface of the island and basin has a smooth concrete finish. A 27.4-m long directional spectral wave maker, which consists of 61 paddles, generates solitary waves for the experiment. Wave absorbers at the three sidewalls reduce reflection in the basin.

Figure 6.4 Schematic sketch of the conical island experiment. (a) Perspective view. (b) Side view (center cross section). ○, gauge locations.
The experiment covers the water depths $h = 0.32$ and 0.42 m and the solitary wave heights $A/h = 0.05$, 0.1 and 0.2. The present study considers the smaller water depth $h = 0.32$ m, which provides a more critical test case for the non-hydrostatic model. In the computation, the solitary wave is generated from the left boundary with the measured initial wave heights of $A/h = 0.045$, 0.096, and 0.181. These measured wave heights, instead of the target wave heights $A/h = 0.05$, 0.1, and 0.2 in the laboratory experiment, better represent the recorded data at gauge 2 and thus the incident wave conditions to the conical island. The radiation condition is imposed at the lateral boundaries to model the effects of the wave absorbers. We use $\Delta x = \Delta y = 5$ cm, $\Delta t = 0.01$ sec, and a Manning’s roughness coefficient $n = 0.016$ for the smooth concrete finish according to Chaudhry (1993). Since wave breaking occurred during the laboratory experiment, we use the upwind flux approximation in the computation to model the processes.

The solution obtained with the momentum-conserved advection scheme provides an illustration of the solitary wave transformation around the conical island. Figure 6.5 shows the results when the wave reaches the maximum elevation on the front face of the island and 2 sec afterward. Because the celerity increases with the wave height, the arrival time of the solitary wave is about 1.4 sec apart for the wave height range considered. The three test cases show similar wave processes despite the difference in amplitude. The results show refraction and trapping of the solitary wave over the island slope. The left panels of Figure 6.6 shows the trapped waves from the two sides superpose with the diffracted wave on the leeside of the island. Wave breaking occurs locally for $A/h = 0.096$ and everywhere around the island for $A/h = 0.181$ according to Titov and Synolakis (1998). This reduces the subsequent runup on the leeside of the island. The right panels shows the free surface 2 sec later when the trapped waves have
Figure 6.5 Wave transformation in front of the conical island. (a) $A/h = 0.045$. (b) $A/h = 0.096$. (c) $A/h = 0.181$. 
Figure 6.6 Wave transformation on the leeside of the conical island. (a) $A/h = 0.045$. (b) $A/h = 0.096$. (c) $A/h = 0.181$. 
passed each other and continue to wrap around to the front. Munger and Cheung (2008) reported similar trapped waves around the Hawaiian Islands generated by the 2006 Kuril Islands Tsunami.

The free surface is rather smooth with indistinguishable frequency dispersion before the wave wraps around the island. As the solitary wave travels down the basin, high-frequency dispersive waves become evident around the island especially on the leeside. The test case with $A/h = 0.181$ provides a vivid depiction of the generation and propagation of the dispersive waves. Figures 6.5(c) and 6.6(c) show the generation of the first group of dispersive waves as the trapped waves wrap around the island and collide on the leeside. After the collision, the second group of dispersive waves is generated due to energy leakage from the two trapped waves that continue to wrap around to the front. The interaction of the first and second groups of dispersive waves generates a mesh-like wave pattern behind the island. These high-frequency dispersive waves provide an explanation for the 3 to 5-min edge waves recorded on the south shore of Oahu after the 2006 Kuril Islands Tsunami that a nonlinear shallow-water model cannot reproduce even with a 10-m computational grid (Bricker et al., 2007).

A number of gauges recorded the transformation of the solitary wave around the conical island. Figure 6.7 shows the time series of the solutions with and without the momentum-conserved advection scheme and the measured free surface elevations at selected gauges. With reference to Figure 6.4, gauges 2 and 6 are located in front of the island and 9, 16, and 22 are placed just outside the still waterline around the island. These gauges provide sufficient coverage of the representative wave conditions in the experiment. The measured data at gauge 2 provides a reference for adjustment of the timing of the computed waveforms. Both solutions show excellent agreement with the measured time
Figure 6.7 Time series of surface elevations at gauges around a conical island. (a) $A/h = 0.045$. (b) $A/h = 0.096$. (c) $A/h = 0.181$. ○, laboratory data from Briggs et al. (1995); (red), non-hydrostatic with momentum-conserved advection; - - - - (blue), without momentum-conserved advection.

series including the depression following the leading wave that was not adequately reproduced in previous studies. The momentum-conserved advection scheme reasonably describes the phase of the peak, but slightly overestimates the leading wave amplitude at gauges 9 and 22 as the wave height increases. Without the momentum conservation, the model generally reproduces the leading wave amplitude except at gauge 22 with $A/h = 0.181$, where the model fails to fully capture the energetic breaking wave on the leeside of the island. This reiterates the importance of the momentum-conserved advection scheme in capturing breaking waves.
Figure 6.8 shows comparisons of the measured and computed inundation and runup around the conical island. Both solutions are almost identical and show good agreement with the laboratory data. The momentum-conserved advection produces better estimations at the lee flank of island around 90° ~ 160°, where the runup is lowest. For the results presented in Figures 6.7 and 6.8, both solutions are comparable or slightly better than the extended Boussinesq solutions of Chen et al. (2000) and Lynett et al. (2002) that use empirical relations with adjustable coefficients to describe wave breaking. Most of the previous studies neglected friction in this numerical experiment (Liu et al., 1995a; Titov and Synolakis, 1998; Chen et al., 2000; and Lynett et al., 2002). A test of the model with n = 0.0 gave very similar results as n = 0.016. As pointed out in Liu et al. (1995a), the computed results are not sensitive to the surface roughness coefficient due to the steep 1:4 slope of the conical island. The overall agreement between the computed results and laboratory data indicates the capability of the present model to estimate wave transformation, breaking, and inundation in the two horizontal dimensions.

Figure 6.8 Inundation and runup around a conical island. (a) A/h = 0.045. (b) A/h = 0.096. (c) A/h = 0.181. ○, laboratory data from Briggs et al. (1995); --- (red), non-hydrostatic model with momentum-conserved advection; ---- (blue), without momentum-conserved advection.
CHAPTER 7
THE 2009 SAMOA TSUNAMI

The objective behind the development of NEOWAVE is to have a model that can describe the tsunami evolution process from generation, propagation to runup with due consideration to dynamic seafloor deformation, wave dispersion, wave breaking, and bore propagation at appropriate resolution. The 2009 Samoa Tsunami affected regions with steep and irregular offshore bathymetry as well as extended shallow fringing reefs. The well-recorded event provides a critical test case to validate NEOWAVE for real-field tsunami modeling.

7.1. Model Setup

The Samoa earthquake occurred near the Tonga trench on 29 September 2009 at 17:48:10 UTC. The US Geological Survey (USGS) determined the epicenter at 15.509°S 172.034°W and estimated the moment magnitude M_w of 8.1. Figure 7.1 shows the rupture configuration and the locations of the water-level stations in the region. The main energy of the resulting tsunami propagated toward Tonga and American Samoa. The tide gauge in Pago Pago Harbor, Tutuila (American Samoa), and the DART buoys 51425, 51426, and 54401 surrounding the rupture area recorded clear signals of the tsunami. The tsunami arrived at mid tide and produced maximum runup of 17.6 m and detrimental impact on Tutuila. The rugged, volcanic island sits on a shallow shelf of less than 100 m deep covered by mesophotic corals (Bare et al., 2010). The insular slope is steep with gradients up to 1:2 on the west side and drops off abruptly to over 3000 m depth in the surrounding ocean. Several field survey teams recorded and documented the tsunami runup and inundation around Tutuila and provided useful data for model validation (Fritz et al., 2009; Koshimura et al., 2009; and Jaffe et al., 2010).
We reconstruct the 2009 Samoa Tsunami from its generation at the earthquake source to runup at Pago Pago Harbor with four levels of nested grids. Figure 7.1(a) shows the coverage of the level-1 grid, which spreads across the south-central Pacific at 1-min ($\approx$ 1800-m) resolution. An open boundary condition allows radiation of tsunami waves away from the domain. Figure 7.2 shows the original and the smoothed bathymetry at the next three levels. The level-2 grid covers American Samoa down to the 4000-m depth contour at 7.5-sec ($\approx$ 225-m) resolution to capture wave transformation around the island group. The level-3 grid resolves the shelf and steep bathymetry down to the 1500-m depth contour around Tutuila at 1.5 sec ($\approx$ 45 m) and provides a transition to the level-4 grid, which covers Pago Pago Harbor at 0.3 sec ($\approx$ 9 m) resolution for computation of inundation as well as tide gauge signals. A Manning’s coefficient $n = 0.035$ describes the
Figure 7.2 Coverage of levels 2, 3, and 4 computational domain. (a) Original bathymetry and topography. (b) Smoothed data with depth-dependent Gaussian function. ○, Pago Pago tide station.

Surface roughness in the near-shore seabed with fringing reefs according to Bretschneider et al. (1986). The momentum-conserved advection scheme is used at the level-4 grid, where flow discontinuities associated with wave breaking and bore formation would otherwise cause volume loss and numerical dissipation.
The digital elevation model is derived from a blended dataset of multiple sources. The 0.5-min ($\approx 900$-m) General Bathymetric Chart of the Oceans (GEBCO) from the British Oceanographic Data Centre (BODC) provides the bathymetry for Pacific Ocean. The Coastal Relief Model from National Geophysical Data Center (NGDC) covers the American Samoa region and Tutuila at 3 and 0.3333 sec ($\sim 90$ and 10 m) resolution respectively. Embedded in the NGDC dataset are multibeam and satellite measurements around Tutuila and high-resolution LiDAR (Light Detection and Ranging) survey data at Pago Pago Harbor. We have converted the datasets to reference the WGS 84 datum and the mean-sea level (MSL). The Generic Mapping Tools (GMT) interpolates the data to produce the computational grids. The original data in Figure 7.2(a) shows fine details of the seafloor. Smoothing of the bathymetry data is sometime necessary to ensure a converging solution from the non-hydrostatic model. Figure 7.2(b) shows the processed data with the depth-dependent Gaussian function. The proposed scheme removes fine features in deep water that should have little effect on tsunami propagation, but retains the near-shore details important for inundation computation.

### 7.2 Tsunami Generation

The present non-hydrostatic model utilizes the vertical velocity to describe dispersive waves. This also facilitates modeling of tsunami generation through dynamic deformation of the seafloor due to earthquake rupture. USGS analyzed the rupture processes of the 2009 Samoa Earthquake using the finite fault inverse algorithm of Ji et al. (2002) and estimated the fault parameters such as depth, orientation, and slip over 420 subfaults of 6 km by 6 km each. The analysis provides the rupture initiation time and rise time for 249 subfaults with seismic moment of over $10^{25}$ dyne-cm.
The planar fault model of Okada (1985) describes the ground surface deformation in terms of the depth, orientation, and slip of a rectangular fault as shown in Figure 7.3. The deformation is a linear function of the slip and dimensions of the fault. Superposition of the planar fault solutions from the subfaults gives the time sequence of the vertical displacement of the seafloor as

\[ \eta(\lambda, \phi, t) = \sum_{i=1}^{n} \eta_i(\lambda, \phi) f_i(t) \]  \hspace{1cm} (7.1)

in which

\[ f_i(t) = \begin{cases} 0 & \text{if } t < t_i \\ \frac{t - t_i}{\tau_i} & \text{if } t_i \leq t \leq (t_i + \tau_i) \\ 1 & \text{if } t > (t_i + \tau_i) \end{cases} \]  \hspace{1cm} (7.2)

where \( n = 249 \) is the number of subfaults for the 2009 Samoa Earthquake, \( \eta_i \) is the vertical ground surface deformation associated with rupture of subfault \( i \) from Okada (1985), and \( t_i \) and \( \tau_i \) are the corresponding rupture initiation time and rise time from the
USGS finite fault solution. The source time function (7.2) defines a linear motion of the slip at each subfault to approximate the rupture process (Irikura, 1983).

The USGS finite fault solution shows an average rise time of 3.4 sec for the subfault movement and a rupture duration of 94 sec for the entire event. Figure 7.4 shows the rupture, seafloor deformation, and free surface elevation during the tsunami generation process. The rupture starts at the epicenter located on the north side of the fault and propagates toward south. Despite the granularity of the finite fault model, the seafloor deformation is rather continuous due to the depth of the fault below the ground surface. The generation of tsunami waves from the seafloor deformation occurs simultaneously with the propagation of the energy away from the source. In addition, the present approach transfers both the kinetic and potential energy from the seafloor to the water. This results in different seafloor deformation and surface wave patterns in contrast to the static deformation approach. The earthquake releases most of the energy in 30 sec and the subsequent rupture has minimal effect on the seafloor deformation. By the end of the rupture at $t = 94$ sec, the tsunami has propagated over a considerable distance with a leading depression toward the Samoa Islands.
Figure 7.4 Time sequence of rupture and tsunami generation. (a) Slip distribution. (b) Sea floor deformation. (c) Surface elevation. ———, uplift contours at 0.2-m intervals; - - - - -, subsidence contours at 1.0-m intervals.
7.3 Surface Elevation and Runup

Tsunami energy propagation is directional with the majority perpendicular to the faultline. Figure 7.5 shows the tsunami wave amplitude over South Central Pacific with a maximum value of 1.8 m over the deep ocean. The main energy propagates toward Tonga in the west and American Samoa in the east with significant impacts to Tutuila, where the tide gauge in Pago Pago Harbor recorded a strong signal of the tsunami. The three DART buoys, which are located off the main energy beams, also recorded clear signals.

Figure 7.5 Maximum surface elevation for the 2009 Samoa Tsunami. ○ (white), water level stations; ○ (red), epicenter of the 2009 Samoa Earthquake; ·······, rupture area.
Pago Pago Harbor is an L-shape embayment with fringing reefs along its shores. The reefs converge at the west end of the harbor forming an extended shallow flat favorable to bore or hydraulic jump formation (Roeber et al., 2010). The water over the 1000-m long reef flat is approximately 10 m deep and the depth increases to 30 m over a distance.
of 500 m from the reef edge. Figure 7.6 shows a series of the computed surface elevation in the inner harbor that corroborates witness accounts during the first wave (Koshimura et al., 2009). The water begins to withdraw approximately 20 min after the earthquake exposing the shallow reef flats along the shores. The first positive wave, which arrives shortly afterward, reaches a surface elevation of 2 m in the inner harbor and floods the low-lying coastal areas on the north and south sides. The wave develops a sharp surface gradient over the reef flat around $t = 26$ min and transforms into a surge as it hits dry land reaching a maximum elevation of 8.14 m and producing extensive damage in the area. A hydraulic jump develops at the reef edge around $t = 31.0$ min as the floodwater returns to the harbor. The wave retreats faster than the receding floodwater resulting in accumulation of water on land and a waterfall over the reef edge until the next wave arrives.

Figure 7.7 shows a comparison of the recorded and computed waveforms and spectra at the Pago Pago tide gauge and the three DART buoys. The computed results show good agreement of the arrival time, amplitude, and frequency content with the measurements. The model reproduces the initial negative wave at the tide gauge and captures the distinct 11 and 18-min oscillations at Pago Pago Harbor. Some discrepancies of the initial waveforms are evident at the DART buoys. The model over-estimates the amplitude of the leading depression at DART 51425 and cannot reproduce the polarity at DART 51426 and 54401. However, the amplitude recorded at these buoys is less than 4% of the 1.8-m maximum amplitude in the open ocean to the east of the source. The error might be attributed to the details of the source mechanism or the assumed location and predefined strike and dip angles of fault in the USGS finite fault solution that become noticeable off the main energy beams. The overall source mechanism is deemed accurate as it reproduces the observations and strong tsunami signals at Pago Pago Harbor. In fact, the
model results at DART 51426 and 54401 reproduce the high-amplitude reflected waves from Tutuila arriving at 1.6 hours and 3.2 hours after the earthquake.

Pago Pago Harbor is 50 m deep, while the 100-m deep outside embayment is sheltered by barrier reefs along the edge of the insular shelf as shown in Figure 7.8(a). The configuration of the harbor and embayment is prone to trapping of long waves and the two distinct peaks in the amplitude spectrum are indicative of resonance. Following the method of Munger and Cheung (2008) based on Fast Fourier Transform, we extract the

![Figure 7.7 Time series and spectra of surface elevations at water level stations.](image)

Figure 7.7 Time series and spectra of surface elevations at water level stations. —— (black), recorded data; —— (red), computed data.
Figure 7.8 Detailed bathymetry and resonance modes around Pago Pago Harbor. (a) bathymetry. (b) First mode at 18 min. (c) Second mode at 11 min. (d) Third mode at 9.6 min. —— (grey), depth contours at 25-m intervals; - - - (grey), depth contours at 500-m intervals.
oscillation modes of the modeled tsunami waves. The spectral analysis depicts three resonance modes at 9.7, 11, and 18 min period in Pago Pago Harbor that are plotted in Figure 7.8. The first resonance at 18 min extends from the embayment into Pago Pago Harbor. The resonance mode shows a node along the edge of the insular shelf and an antinode at the tip of the harbor. The second resonance mode at 11 min covers the same region with two nodes located at the harbor entrance and the edge of the insular shelf. The third mode at 9.7 min, with a relatively lower energy level in this event, has an additional node in the outside embayment. The combined effect of the first two resonance modes provides an explanation for the significant inundation and property damage despite the well-sheltered location of the harbor.

The results presented so far have been based on the smoothed bathymetry in Figure 7.2 (b). We compute the runup and inundation at level 4 with both the smoothed and original bathymetry to assess the effectiveness of the depth-dependent smoothing scheme. Figure 7.9 compares the computed runup and inundation around Pago Pago Harbor with the measurements from Fritz et al. (2009) and Koshimura et al. (2009). The two sets of computed runup results are almost identical thereby verifying the proposed smoothing scheme that serves to improve model convergence without appreciable alteration of the model results. The smoothing of the bathymetry results in a 10% reduction of the number of iterations in the non-hydrostatic solver that is relatively minor because of the lack of small-scale, steep bottom features in the level-4 grid. In some situations, smoothing is necessary to obtain converging non-hydrostatic solutions. The model reproduces the overall pattern of the runup along the shorelines. The minor discrepancies between the computed results and measurements are primarily due to errors in topographic data and the difficulties in modeling inundation in the built environment around Pago Pago Harbor. The large runup at the tip and low values in the outer harbor show strong correlation with
the three resonance modes. Such oscillation patterns also provide an explanation of the local amplification and the disparate property damage along the coastlines of Tutuila.

Figure 7.9 Runup and inundation at inner Pago Pago Harbor. —— (white), recorded inundation; ○(white): recorded runup; ○(blue): recorded flow depth plus land elevation; —— (red), solution with smoothed bathymetry; - - - - (blue), solution with non-smoothed bathymetry; —— (black), coastline; —— (grey), depth contours at 10-m intervals; - - - - (black), computed runup projection.
CHAPTER 8
CONCLUSIONS AND RECOMMENDATIONS

This dissertation has demonstrated the versatility and robustness of a depth-integrated, non-hydrostatic model for tsunami research and impact assessment. The key feature in the formulation is the decomposition of the pressure into hydrostatic and non-hydrostatic components and the introduction of a linear vertical velocity in conjunction with the non-hydrostatic pressure. This allows modeling of dynamic seafloor deformation in tsunami generation and wave dispersion effects during the evolution processes. The hydrostatic component is equivalent to a nonlinear shallow-water model with an explicit scheme. An implicit scheme provides the non-hydrostatic pressure through the three-dimensional continuity equation. The semi-implicit, finite difference model describes flow discontinuities associated with breaking waves and bore development using the momentum-conserved advection scheme. An upwind flux scheme extrapolates the free surface elevation instead of the flow depth in the computation of the continuity equation and the momentum-conserved advection to improve model stability. A depth-dependent Gaussian function improves the convergent rate in the implicit solution of the non-hydrostatic pressure.

The lower-order spatial derivatives in the governing equations allow implementation of a grid-nesting scheme for the shock-capturing, dispersive model. A two-way nesting scheme exchanges the horizontal velocity, surface elevation, and non-hydrostatic pressure between an inner and an outer grid at every outer grid time step. The use of both the velocity and surface elevation in the nesting scheme is necessary to implement the momentum-conserved advection scheme as well as the upwind flux scheme across the inter-grid boundaries. A Dirichlet boundary condition for the non-hydrostatic pressure ensures wave dispersion is continuous across the inter-grid boundaries. This nested-grid
enables the use of non-rectangular computational domains that can adapt to bathymetric contours and features for optimization of computational time. The model in the spherical coordinate system accounts for the earth’s curvature in basin-wide tsunami propagation and yet is flexible enough to describe detailed wave transformation and runup at a coastal region.

The present depth-integrated dispersive model shows better dispersion characteristics in comparison to the classical Boussinesq equations. Numerical experiments of solitary wave propagation in a channel, sinusoidal wave transformation over a submerged bar, and $N$-wave transformation and runup in Monai Valley verify the dispersion characteristics and the grid-nesting scheme. The momentum-conserved advection scheme captures flow discontinuities associated with breaking waves as bores and hydraulic jumps and reproduces the results in the plane-beach and conical island runup experiments. The model can describe wave breaking over steep slope, but underestimates the runup on gentle slope without the momentum-conserved advection, when breaking becomes more energetic. The present model provides comparable results with existing depth-integrated non-hydrostatic models in wave propagation and transformation, and similar or slightly better estimates than extended Boussinesq models in wave transformation and runup in the test cases considered. The upwind flux approximation in the continuity equation has little effect on wave propagation, while it is essential in providing stable solutions especially when energetic wave breaking occurs.

The present model is applied to reconstruct the 2009 Samoa Tsunami from the USGS finite fault solution. The model reproduces resonance and flow discontinuities and provides an explanation of the observed tsunami behaviors and impacts in Pago Pago Harbor. The computed surface elevations at the DART buoys and the Pago Pago Harbor tide gauge as well as the runup around the harbor show very good agreement with
recorded data. The computed results with and without smoothing of the bathymetry are indistinguishable demonstrating the effectiveness of the proposed smoothing scheme with the depth-dependent Gaussian function.

Numerical models for tsunami flood hazards need to deal with wave dispersion in basin-wide propagation as well as flow discontinuities due to wave breaking nearshore, but at the same time, must be articulate, stable in dealing with flows over complex topography and efficient for large computational problems. The shock-capturing, non-hydrostatic model with the grid-nesting scheme appears to satisfy these requirements for practical application. The computing time depends on the grid size and number of iterations in the non-hydrostatic solver. For the results presented in this dissertation, the computing time is about 1.5 to 3.5 times in comparison to the hydrostatic solution. The turn-around time can be reduced through parallel computing. The model results are very stable and do not show any spurious oscillations even with the most energetic wave breaking conditions. Among the existing dispersive models, the present model utilizes the simplest dispersive term, which is only the first derivative of the non-hydrostatic pressure. This would be one reason for the stability achieved by the present model. The use of the upwind flux approximation of the surface elevation in the continuity and momentum equations further improves the model stability and becomes essential with wave breaking.

The work described in this dissertation has developed an alternative direction in tsunami modeling with ample opportunities for further research and development. These include enhancement of the dispersion characteristics for modeling of coastal processes, parallelization of NEOWAVE for improvement of computational efficiency, extension of the dynamic seafloor deformation to include horizontal movement, coupling with a three-dimensional hydrodynamics model for landslide-generated tsunamis, and coupling with spectral wave models for storm surge prediction. Future applications include tsunami
inundation mapping for Hawaii including the Northwest Hawaiian Islands, American Samoa, the US Gulf coasts, Puerto Rico, and Chile, paleotsunami modeling and impact assessment for Western Samoa, as well as storm surge modeling for Hawaii and the US east coast. The present theoretical and numerical formulations, which build on the nonlinear shallow-water equations, can be implemented in most commonly used long-wave models to describe generation, dispersion, breaking, and runup of tsunami waves.
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